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Almost pure phase mass matrices from six dimensions

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Abstract

A model of quark masses and mixing angles is constructed within the framework of two large extra compact dimensions. A "democratic" almost pure phase mass matrix arises in a rather interesting way. This type of mass matrix has often been used as a phenomenologically viable ansatz, albeit one which had very little dynamical justification. It turns out that the idea of large extra dimensions provides a fresh look at this interesting phenomenological ansatz as presented in this paper. Some possible interesting connections to the strong CP problem will also be presented.

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1. Introduction

The question of the origin of fermion mass hierarchy, mixing angles and CP violating phase is one of the most outstanding problems in particle physics. There have been numerous attempts to study this problem, some of which are more theoretical in nature while others are more phenomenological. However, it is generally agreed that the final word is far from being said. Furthermore, it is also agreed that the solution, whatever it might be, is to be found outside of the Standard Model (SM).

In all of these studies, the phenomenological-ansatz approach is much more modest in scope. Starting with some simple assumption about the form of the mass matrix whose theoretical justification is yet-to-be-determined, one could fit quark masses and mixing angles. One of such approaches is particularly appealing: the pure phase mass matrix

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(PPMM) [1,2]. This particular ansatz is based on a simple assumption that there is a single and unique Yukawa coupling for each quark sector and that the 3×3 mass matrix takes the form $\mathcal{M} = g_Y(v/\sqrt{2})\{\exp(i\theta_{ij})\}$, where $i, j = 1, 2, 3$. This kind of mass matrices belongs to a class of the so-called "democratic mass matrices" (DMM) [3]. The pure phase mass matrix is attractive in that the hierarchy of masses is governed by a single Yukawa coupling in the limit where all phases vanish. A realistic hierarchy comes about when the phases, which are treated as small perturbations, are put back in. Although it is conceptually attractive, no attempt was made to justify its underlying assumption. Earlier works on trying to model the pure phase mass matrix relied entirely on the framework of four-dimensional field theories. Although there are a number of useful lessons that can be learned from this mode of thinking, one is sometimes faced with more questions than answers.

On another front, there has been important conceptual developments in the last few years related to a possible existence of large extra dimensions [4,5]. Not only does this concept force us to rethink about notions such as the question of what the ultimate fundamental scale of nature might be, it also inspires us to reformulate some of the longstanding problems in particle physics such as the origin of fermion masses and mixings. The hierarchy of masses has been reexamined recently within the framework of large extra dimensions, and new interesting ideas have emerged such as the notion of "thick branes" and the localization of various fermions inside these branes [6]. This localization can be accomplished by a domain wall inside the brane. This gave rise to the idea of the strength of the Yukawa coupling (which is proportional to the mass of the fermion) as being the overlap of the wave functions of the localized fermions. As stated in Ref. [6], it is easy to think of the reason why some fermions are heavy and some are light: the heavy ones have large overlap and the light ones have small overlaps. There has been some works done along that line in order to explain the fermion mass hierarchies. Most of these works made use of the size of the wave function overlaps to discuss the fermion mass problem.

Whatever various scenarios might be, the common important elements which transpired from these works are basically the locations of the domain walls and the size of the wave function overlaps. In fact, many of the physics results will depend on the actual placements of the domain walls along the extra dimensions.

Our approach in this paper is as follows: for each fermion sector (e.g., the up and down quark sectors), there is a universal overall mass scale whose Yukawa coupling strength is determined by the size of the overlap. This gives rise to a democratic mass matrix whose elements are all equal to unity, apart from a common mass scale factor multiplied by an effective Yukawa coupling. All that is needed is to localize all the left-handed fermions at one location, regardless of family indices, and all the right-handed fermions at another location along the fifth dimension inside the thick brane, and, in addition, to endow the fermions with a permutation symmetry. Unfortunately, it is well known that this kind of matrix does not work: one obtains *one non-zero* mass eigenvalue and *two zero* eigenvalues. The matrix $\{1\}$ has to be replaced by another quasi-democratic one of the form such as $\{\exp(i\theta_{ij})\}$, for example. The mass hierarchy which arises *within* each sector is due, in our scenario, to the introduction of a sixth dimension and a thick brane along it. The introduction of "family" domain walls at different locations inside this thick brane generate different phases for different families. It will be seen that it is these phase differences which give rise to the *pure phase* mass matrix or, as we shall see, an almost-pure phase mass matrix. We would like to stress that, although a permutation symmetry was used, the results obtained are purely *geographical* in nature.

We would like to make the following remark. Our model will contain a certain number of parameters that need to be fixed phenomenologically. However, what we present here is a new perspective on an old problem which, hopefully, can give further insights which might be useful for future investigations. What we are doing here is to try to rephrase the origin of quark mass hierarchy (and eventually that of the leptons as well) and CP phase in a completely new context: that of the Compact Extra Dimensions (CED). We will show below that the appearance of the phases in the mass matrices, a crucial element in their construction, appear rather “naturally”. From this point of view, it appears to be a definite conceptual advantage of the CED scenario.

One remark is in order here concerning the introduction of a sixth dimension. It is well known that, with just one extra compact dimension, the fundamental 5-dimensional Planck scale cannot be of the order of a few TeV or so, for it will introduce deviations to the inverse square law on astronomical distances. Recent gravity experiments [7] down to a millimeter or so put a lower bound of around 3 TeV on the $4+n$ Planck scale for the case of $n=2$ (with equal compactification radii). This fact, of course, was not the one motivating us in introducing a sixth dimension. It is rather the natural way in which phase differences appear between different fermions eventually giving rise to a pure phase mass matrix which motivated us.

The organization of the paper is as follows. First we review various features of fermions in five dimensions, including, for instance, the concept of fermion localization. We then show how, with a rather simple assumption, a democratic mass matrix appears. Next, we introduce fermions in six dimensions and show how phase differences appear, and how one can construct an (almost) pure phase mass matrix from this result. In this construction, “family” domain walls are introduced and it is shown that their small separations along the sixth dimension are responsible for the aforementioned phase differences. Unlike what happens along the fifth dimension, the fermion wave functions are not of the localizing type but are rather oscillating. We will then discuss how Hermitian and non-Hermitian pure phase mass matrices arise. Finally, we will discuss some possible connections to the strong CP problem [8].

2. Fermions in 5 dimensions and democratic mass matrix

2.1. A review

In this section, we will review some aspects of fermions in five dimensions which have support $[0, L]$ along the fifth dimension. In other words, we are discussing a “thick brane” of thickness L. This discussion serves two purposes: to set the notations and to lead to the democratic mass matrix.

We will adopt the effective field theory approach of Refs. [10,11]. This approach has the merit of being relatively simple and transparent as far as the physics is concerned. We first summarize below what has been done for the case of one flavor of fermions, without and with a background scalar field.

To set the notations straight, the 4-dimensional coordinates will be labeled by x^μ with $\mu = 0, \ldots, 3$ while the fifth coordinate will be labeled by y. We start out with a free Dirac spinor of $SO(4,1)$ which has four components, ψ. The gamma matrices are γ^μ and $\gamma_y = i\gamma_5$. The free Dirac Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\left(i\partial\!\!\!/ + i\gamma_y \frac{\partial}{\partial y}\right)\psi, \\ &= \bar{\psi}\left(i\partial\!\!\!/ - \gamma_5 \frac{\partial}{\partial y}\right)\psi. \end{aligned} \tag{1}$$

The above Lagrangian has the following Z_2 symmetry: $\psi(x, y) \to \Psi(x, y) = \pm\gamma_5\psi(x, L - y)$. When this symmetry is combined with the periodic boundary condition: $\psi(x, y) = \Psi(x, L + y) = \psi(x, 2L + y)$, one obtains: $\psi(x, -y) = \Psi(x, L - y) = \pm\gamma_5\psi(x, y)$ and $\psi(x, L + y) = \Psi(x, y) = \pm\gamma_5\psi(x, L - y)$, which shows that $y = 0, L$ are fixed points. One can subsequently define the chiral components of ψ by using the usual operators $P_{R,L} = (1 \pm \gamma_5)/2$, with $\psi_+ = P_R\psi$ and $\psi_- = P_L\psi$, with $\gamma_5\psi_\pm = \pm\psi_\pm$. The previous symmetry and boundary conditions are what usually referred to in the literature as compactification on an S_1/Z_2 orbifold. One can have fermions which have the symmetry $\psi(x, y) \to \Psi(x, y) = +\gamma_5\psi(x, L - y)$, and those which have $\psi(x, y) \to \Psi(x, y) = -\gamma_5\psi(x, L - y)$.

For simplicity, we shall discuss the case $\psi(x, y) \to \Psi(x, y) = +\gamma_5\psi(x, L - y)$ below. This corresponds to the case where only *right-handed* zero modes survive in the brane, as shown below. For the other situation, $\psi(x, y) \to \Psi(x, y) = -\gamma_5\psi(x, L - y)$, only the *left-handed* zero modes survive inside the brane, as one can easily check.

Zero modes residing in the brane are supposed to be independent of the extra coordinate, y in this case. From the above discussion, one can see that ψ_- vanishes at the fixed points, and hence there is no zero mode for ψ_-. The only non-vanishing zero mode is ψ_{0+}. This can also be seen explicitly by writing

$$\psi_{M+}(x, y) = \psi_{M+}(x)\xi_{M+}(y), \tag{2a}$$

$$\psi_{M-}(x, y) = \psi_{M-}(x)\xi_{M-}(y), \tag{2b}$$

for a mode of mass M. From the explicit solutions for ξ as given in Ref. [10], one can again see that there is only one chiral zero mode inside the brane. Four-dimensional chirality is seen to arise from the symmetry and boundary conditions. The chiral zero mode ψ_{0+} is *uniformly* spread over the fifth dimension y. To localize ψ_{0+} at specific points along y inside the brane, the use of domain walls have been suggested by Refs. [6,10]. To this end, a background scalar field, Φ, is introduced. The Lagrangian is given by

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - \gamma_5 \frac{\partial}{\partial y} - f\Phi\right)\psi + \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - \frac{1}{2}\partial_y\Phi\partial_y\Phi - \frac{\lambda}{4}\left(\Phi^2 - V^2\right)^2. \tag{3}$$

The symmetry and boundary conditions on Φ are now: $\Phi \to \widetilde{\Phi}(x, L - y) = -\Phi(x, y)$; $\Phi(x, -y) = \widetilde{\Phi}(x, L - y) = -\Phi(x, y)$ and $\Phi(x, L + y) = \widetilde{\Phi}(x, y) = -\Phi(x, L - y)$. It can then be seen that ϕ vanishes at the orbifold fixed points: $y = 0, L$. As discussed in Ref. [10],

Φ has a minimum energy configuration: $\langle \Phi(x,y)\rangle = \phi(y)$, with $\phi(0)=\phi(L)=0$. From the modified equations for $\xi_{M\pm}$ with an added term $f\phi(y)$, one can easily see the localization of the zero mode, namely,

$$\xi_{0+}(y) = ke^{-s(y)}, \qquad \xi_{0-}(y) = 0, \tag{4}$$

where

$$s(y) = f\int_0^y dy'\,\phi(y'). \tag{5}$$

As pointed out by Ref. [10], the chiral zero mode, $\xi_{0+}(y)$, is now localized either at $y=0$ or $y=L$ depending on the sign of $f\phi(y)$.

As in Ref. [6], the special choice $f\phi(y) = 2\mu^2 y$ which makes the operators $a = \partial_y + f\phi(y)$ and $a^\dagger = -\partial_y + f\phi(y)$ behave like the annihilation and creation operators of a Simple Harmonic Oscillator (SHO), the normalized wave function for the chiral zero mode $\xi_{0+}(y)$ takes on the familiar form $\xi_{0+}(y) = (\sqrt{\mu}/(\pi/2)^{1/4})\exp(-\mu^2 y^2)$. One clearly notices the localization of $\xi_{0+}(y)$ at $y=0$. Another way of describing this phenomenon is the fact that ϕ has a kink solution of the form $V\tanh((\lambda/2)^{1/2}Vy)$ which basically traps the fermion to a domain wall of size $((\lambda/2)^{1/2}V)^{-1}$ [12].

The next question concerns the possibility of localizing the chiral zero mode at some other location than the one at the orbifold fixed points. Ref. [6] has proposed to change the Yukawa interaction $\bar{\psi}(f\phi(y))\psi$ to $\bar{\psi}(f\phi(y)-m)\psi$ so that the wave function of the chiral fermion field is now localized at the *zero* of $f\phi(y)-m$ instead of $f\phi(y)$. With the SHO approximation, this zero would be at $y = m/2\mu^2$. However, in order to be compatible with the Z_2 symmetry of the Lagrangian, as shown in Eq. (3), one should also require a "mass reversal" $m \to -m$ simultaneously with the Z_2 transformations. This is the assumption we will be making in this manuscript. (Another approach is given in Ref. [10].)

As emphasized by Ref. [6], different massless chiral fermions can be localized on different slices along y, inside the thick brane. These locations are determined by the zeros of $f\phi - m_i = 0$. Within the SHO approximation, the wave functions are given by $(\sqrt{\mu}/(\pi/2)^{1/4})\exp(-\mu^2(y-y_i)^2)$, where $y_i = m_i/2\mu^2$. The interesting idea proposed in Ref. [6] is that the effective Yukawa couplings between SM fermions and SM Higgs scalar, which eventually determines the size of the mass term, are mainly determined by the wave function overlap between the left- and right-handed fermions. Hierarchy of masses then appears to depend on the size of the overlaps.

From hereon, we shall turn our attention to *left-handed* zero modes inside the brane as used in the SM. As we have mentioned earlier, these come from five-dimensional fermions with the Z_2 symmetry $\psi(x,y) \to \Psi(x,y) = -\gamma_5\psi(x, L-y)$.

To prepare the groundwork for our subsequent discussion, let us write down the action in five dimensions of a left-handed fermion, a right-handed fermion, and the Yukawa interactions with a background scalar field, and a SM Higgs field. Following Ref. [6], we will denote quarks in five dimensions by the five-dimensional Dirac fields: (Q, U^c, D^c) and their *left-handed* zero modes by the following Weyl fields: (q, u^c, d^c). Notice that with this notation, a right-handed down quark, for example, will be $\bar{d}^c$. Since we will be dealing in this paper solely with the quark sector, we are not writing down the lepton fields.

This will be dealt with in a subsequent paper. The SM transformations of the above fields are self-evident by the use of these notations. In addition, one also introduces two sets of scalar fields: a SM singlet background scalar field, ϕ, whose VEV is $\langle \Phi(x,y)\rangle = \phi(y)$, a SM doublet Higgs field $H(x,y)$ whose zero mode $h(x)$ is assumed to be uniformly spread along y inside the thick brane. The 5-dimensional action can be written as

$$S = \int d^5x\, \overline{Q}\big(i\partial\!\!\!/_5 + f\phi(y)\big)Q + \overline{U}^c\big(i\partial\!\!\!/_5 + f\phi(y) - m_U\big)U^c \\ + \overline{D}^c\big(i\partial\!\!\!/_5 + f\phi(y) - m_D\big)D^c + \kappa_U Q^T C_5 H U^c + \kappa_D Q^T C_5 \widetilde{H} D^c, \tag{6}$$

where $C_5 = \gamma_0\gamma_2\gamma_y$. From the above equation, one notices that Q, U^c, D^c are localized at $y_Q = 0$, $y_U = m_U/2\mu^2$, $y_D = m_D/2\mu^2$, respectively. In principle, m_U and κ_U can be different from m_D and κ_D, respectively. However, as we can see below, it is sufficient to have $m_U \neq m_D$ in order for the resulting masses of up and down quarks to be different, even if $\kappa_U = \kappa_D$. Assuming that the zero mode of H is uniformly spread over y inside the thick brane, the 4-dimensional effective action for the Yukawa interaction for the up quark can be written as

$$S = \int d^4x\, \kappa_U q^T(x)h(x)u^c \int dy\, \xi_q(y)\xi_{u^c}(y), \tag{7}$$

and similarly for the down quark. From the form of the wave functions, one obtains the 4-dimensional effective Yukawa couplings for up and down quarks as follows

$$g_{Y,u} = \kappa_U \exp\big(-\mu^2 y_U^2/2\big), \tag{8}$$

$$g_{Y,d} = \kappa_D \exp\big(-\mu^2 y_D^2/2\big). \tag{9}$$

Two remarks can be made concerning Eqs. (8) and (9). First of all, as emphasized by Ref. [6], even if κ's are of order unity, the effective Yukawa couplings can be quite small if $\mu y_{U,D} \gg 1$. Basically, the size of the effective coupling is sensitive to the relative distance between left- and right-handed quarks as compared with the characteristic thickness of the domain walls. The second remark concerns the Yukawa couplings in five dimensions, $\kappa_{U,D}$. In this new framework of large extra dimensions, one has to separate the mechanism which separates $g_{Y,u}$ from $g_{Y,d}$, already at the level of the 5-dimensional action from that which separates $g_{Y,u}$ from $g_{Y,d}$ at an effective field theory level in four dimensions due to different localization points along the extra dimension inside the thick brane. It might happen that the 5-dimensional action has an up–down symmetry in the Yukawa sector which is broken down inside the brane. We shall return to this question at the end of the paper.

2.2. Democratic mass matrix

Let us, for now, concentrate on just one sector, e.g., the up sector. Let us assume that there are three families. The fermion fields in five dimensions that we will be dealing with in this section will be Q and U^c. As we shall see below, in order to obtain the DMM scenario, we will put all the Q's at one location along y inside the thick brane, and all the U^c's at another location. With this simple assumption and the assumption that the SM

Higgs zero mode is uniformly spread inside the thick brane, one can naively obtain the democratic mass matrix mentioned above. However, with the gauge field zero modes also spreading uniformly inside the thick brane, this will give rise to unwanted flavor-changing neutral current (FCNC) operators. A symmetry has to be imposed in order to avoid these FCNCs.

A simple symmetry that one can use is a permutation symmetry among the three families, for both Q and U^c. One can have: $S_3^Q \otimes S_3^{U^c}$, with $Q \to S_3^Q Q$ and $U^c \to S_3^{U^c} U^c$. The background scalar field described earlier $\phi(y)$ is a singlet under the above permutation group. (In this way, one will see that all Q's are localized at one place and all U^c's are localized at another place.) One can now include gauge interactions in the kinetic terms of (6) by making the replacement $\partial\!\!\!/_5 \to D\!\!\!\!/_5$, namely,

$$S_0 = \int d^5x\, \overline{Q}\big(iD\!\!\!\!/_5 + f\phi(y)\big)Q + \overline{U}^c\big(iD\!\!\!\!/_5 + f\phi(y) - m_U\big)U^c \\ + \overline{D}^c\big(iD\!\!\!\!/_5 + f\phi(y) - m_D\big)D^c. \tag{10}$$

It is simple to see that S_0 is invariant under the above permutation symmetry. Eq. (10) also implies that all Q's are localized at one place and all U^c's are localized at another place.

Next, we wish to introduce a Yukawa interaction between the SM Higgs scalar and Q and U^c. First, we notice that a term such as

$$\mathcal{L}_{\text{Yukawa}} = \kappa_U Q^T C_5 H U^c + \text{h.c.}, \tag{11}$$

breaks the permutation symmetry since Q and U^c transform under different groups. If they were to transform under the *same* permutation group, Eq. (11) would be an invariant. However, it would give a mass matrix of the form

$$\mathcal{M} = g_{Y,u}\frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

which is not of the DMM type. It turns out that with $S_3^Q \otimes S_3^{U^c}$, one can construct an *invariant* for each permutation group: $\sum_i Q_i$ for S_3^Q and $\sum_j U_j^c$ for $S_3^{U^c}$ where $i = 1, 2, 3$ and $j = 1, 2, 3$ are family indices. From this, one can construct an invariant action for the Yukawa interaction

$$S_{\text{Yukawa}} = \int d^5x\, \kappa_U \sum_i Q_i^T C_5 H \sum_j U_j^c + \text{h.c.} \tag{13}$$

The effective action in four dimensions can now be written as

$$S_{\text{eff, Yukawa}} = \int d^4x\, \kappa_U \sum_{i,j} q^{T,i}(x) h(x) u^{c,j} \int dy\, \xi_q^i(y)\xi_{u^c}^j(y) + \text{h.c.} \tag{14}$$

Since all the q_i's are located at the same place inside the brane, and similarly for all the u_i^c, the wave function overlap $\int dy\, \xi_q^i(y)\xi_{u^c}^j(y)$ is universal and *independent* of i, j. With this, one can now rewrite Eq. (14) as

$$S_{\text{eff, Yukawa}} = \int d^4x\, g_{Y,u} q^T(x)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} h(x) u^c + \text{h.c.}, \tag{15}$$

where $g_{Y,u}$ is given by Eq. (8), $q^T = (q_1^T, q_2^T, q_3^T)$ and similarly for $u^c(x)$. From Eq. (15), one obtains the democratic mass matrix

$$\mathcal{M} = g_{Y,u}\frac{v}{\sqrt{2}}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}. \tag{16}$$

An important remark is in order here. The universal strength in Eq. (16) depends on, besides the SM quantity $v/\sqrt{2} \sim 175$ GeV, $g_{Y,u}$ which is a product of two factors: the five-dimensional Yukawa coupling, κ, and the overlap of left- and right-handed fermion wave functions. In this scenario and its extension presented below, it is this product that is important, and not simply the size of the overlap.

As we have mentioned above, the above matrix can be brought by a similarity transformation to a form

$$\begin{aligned}\mathcal{M}' &= S\mathcal{M}S^{-1}\\ &= g_{Y,u}\frac{v}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 3\end{pmatrix}.\end{aligned} \tag{17}$$

As one can see above, one needs to move beyond the DMM scenario in order to obtain a more "realistic" mass matrix. This is what we propose to do in the next section.

One might wonder what the distinctive feature a fifth dimension has to give us in regards with the above problem. Could one not obtain a similar result staying in just four dimensions? In principle, the answer is yes. However, it appears more attractive to think that, once q_i are lumped together at one place and u_j^c are lumped at another place, one would obtain the DMM naturally. It is interesting to envision a scenario in which the Yukawa couplings are as universal as the gauge couplings themselves, with the possibility that the effective Yukawa couplings can be different from one another due to the different overlaps between left and right fermions. (Gauge interactions are chirality conserving and, as a result, the effective gauge coupling with the gauge boson zero mode is the same as the original coupling.)

The above discussion carries over to the down sector in a similar fashion. Obviously, although attractive, this kind of democratic mass matrix does not give the correct mass spectrum. An extension of DMM was discussed by Ref. [1], in which, instead of having one's as matrix elements, one has pure phase factors such as $\exp(i\theta_{ij})$. (The diagonal elements can be all unity by a suitable redefinition of the quark phases.) Explicitly, a pure phase mass matrix looks like $\mathcal{M} = g_Y(v/\sqrt{2})(\exp(i\theta_{ij}))$.

To construct a model for PPMM—even for the special case such as a symmetric matrix, one usually requires a rather complicated Higgs structure [2]. That is if one stays in four dimensions. One might wonder if extra dimensions might help in this regards. We have seen above how an additional dimension could help conceptually in obtaining a democratic mass matrix. The question we ask is the following: could pure phases such as $\exp(i\theta_{ij})$ arise from extra dimensions and not from some kind of complicated Higgs sector? In particular, if we keep the Higgs sector to a minimum (one Higgs), this phase cannot come from the Yukawa coupling nor from the VEV of the SM Higgs. We have seen that, in five dimensions, a chiral zero mode has, as a part of its wave function, $\xi(y)$ which behaves, upon being trapped by a domain wall, like $\exp(-\mu^2 y^2)$. As we shall see below, by adding another compact dimension (the sixth one), the phases appear as the overlaps between wave functions of fermions which are "trapped" at different locations along the 6th dimension. What this really means will be explored in the next section.

3. Fermions in 6 dimensions and pure phase mass matrix

Notwithstanding the string theory argument, there might be another simpler motivation for the need of more than one extra spatial dimension: if the fundamental $4+n$ "Planck" scale were of $O(\text{TeV})$ to "solve" the hierarchy problem, and if the n extra dimensions were to be compactified with the same radius R then $n \geqslant 2$ in order for R to be in the submillimeter region as required by the lack of deviation from the ordinary inverse square law down to about 0.2 mm [7]. In our case, the above need is dictated by the desire to build a more "realistic" mass matrix: the so-called pure phase mass matrix or its almost-pure-phase counterpart. (In this construction, we are not concerned about whether or not the ultimate theory contains more than six dimensions.) To this end, we first study the behaviour of fermions in six dimensions, subject to similar boundary conditions as in the 5-dimensional case.

3.1. *Fermions in six dimensions*

The task of this section is to study fermions in six dimensions, with the ultimate aim of obtaining massless chiral fermions in four dimensions.

In order to discuss fermions in six dimensions, we first turn our attention to the representation of gamma matrices for these fermions. Before we begin the discussion, a few remarks concerning spinors in $SO(N)$ are necessary.

We shall be working with the group $SO(5,1)$ that, as we discuss in Appendix A, has two irreducible spinor representations of dimension 4. We shall put ψ_+ and ψ_- into a reducible "Dirac" spinor $\psi = (\psi_+, \psi_-)$. The chiral representation of the gamma matrices for $SO(5,1)$ is shown in Appendix A. The notation for the coordinates will be similar to the five-dimensional case, with the sixth dimension denoted by z, namely, $x_N = (x_0, x_1, x_2, x_3, y, z)$. The free Lagrangian for ψ is now written as

$$\mathcal{L}_\psi = i\bar{\psi}\Gamma^N \partial_N \psi, \tag{18}$$

where $N = 0, 1, 2, 3, y, z$. The metric used in this paper is simply $(-+++++)$. It is useful to see explicitly the Lagrangian written in terms of the components of ψ. For this purpose, we give the explicit forms for Γ_y and Γ_z as can be seen from Appendix A,

$$\Gamma_y = \begin{pmatrix} 0 & -i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix}, \tag{19}$$

$$\Gamma_z = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{20}$$

where γ_5 is the usual matrix encountered in four dimensions and $\mathbb{I}$ is a 4×4 unit matrix. In addition, we also need $\bar{\psi} = \psi^\dagger \Gamma_0 = (\bar{\psi}_-, -\bar{\psi}_+)$. Eq. (18) can now be rewritten as

$$\mathcal{L}_\psi = -i\bar{\psi}_+\gamma^\mu\partial_\mu\psi_+ - i\bar{\psi}_-\gamma^\mu\partial_\mu\psi_- + \bar{\psi}_+\gamma^5\partial_y\psi_+ + \bar{\psi}_-\gamma^5\partial_y\psi_- \\ - i\bar{\psi}_+\partial_z\psi_+ + i\bar{\psi}_-\partial_z\psi_-. \tag{21}$$

As we explain in Appendix A, the 4-dimensional kinetic terms (the first two terms of the above equation) will acquire a plus sign when γ^μ are replaced by $\tilde{\gamma}^\mu$ which are appropriate for the metric $(-+++)$ which is a remnant of the original metric $(-+++++)$. The reader is strongly recommended to consult Appendix A concurrently with this section in order to avoid confusion.

As in the case of the fifth dimension, we will assume that the sixth dimension is compactified on an orbifold S_1/Z_2. ψ is assumed to have support $[0, L_6]$ along the sixth dimension. We first discuss this Z_2 symmetry for free fermions.

From Eq. (18), one can see that the Lagrangian has the following Z_2 symmetry:

$$\psi(x^\alpha, z) \to \Psi(x^\alpha, z) = \Gamma_z\psi(x^\alpha, L_6 - z). \tag{22}$$

With Γ_z given above, this symmetry translates into

$$\psi_+(x^\alpha, z) \to \Psi_+(x^\alpha, z) = \psi_-(x^\alpha, L_6 - z), \\ \psi_-(x^\alpha, z) \to \Psi_-(x^\alpha, z) = \psi_+(x^\alpha, L_6 - z). \tag{23}$$

As with the five-dimensional case, our boundary condition is

$$\psi_\pm(x^\alpha, z) = \Psi_\pm(x^\alpha, L_6 + z) = \psi_\pm(x^\alpha, 2L_6 + z). \tag{24}$$

Again, combining (23) with (24), one obtains

$$\psi_\pm(x^\alpha, -z) = \psi_\mp(x^\alpha, z), \tag{25}$$

$$\psi_\pm(x^\alpha, L_6 - z) = \psi_\mp(x^\alpha, L_6 + z). \tag{26}$$

We immediately recognizes $z = 0, L_6$ to be the fixed points of the orbifold. It is convenient to rewrite $\psi_\pm$ as

$$\psi_\pm = \frac{1}{\sqrt{2}}(\chi \pm \eta). \tag{27}$$

In terms of χ and η, the boundary conditions become

$$\chi(x^\alpha, -z) = \chi(x^\alpha, z), \tag{28}$$

$$\eta(x^\alpha, -z) = -\eta(x^\alpha, z), \tag{29}$$

$$\chi(x^\alpha, L_6 - z) = \chi(x^\alpha, L_6 + z), \tag{30}$$

$$\eta(x^\alpha, L_6 - z) = -\eta(x^\alpha, L_6 + z). \tag{31}$$

From the above boundary conditions, one can see that η *vanishes* at the fixed points $z = 0, L_6$.

As usual, we shall write:

$$\chi_M\big(x^\alpha, z\big) = \chi_M\big(x^\alpha\big)\xi_{\chi,M}(z), \tag{32a}$$

$$\eta_M\big(x^\alpha, z\big) = \eta_M\big(x^\alpha\big)\xi_{\eta,M}(z). \tag{32b}$$

Since the *zero modes* in the "4-brane" are *independent* of z, we have

$$\chi\big(x^\alpha, z\big)_0 = k\chi\big(x^\alpha\big), \qquad \eta\big(x^\alpha, z\big)_0 = 0, \tag{33}$$

where k is a constant. Again, the free fermion wave function for the zero mode is uniformly spread over the 6th dimension. We now investigate the effect of a coupling with a background scalar field having a kink solution.

For the discussion which follows, it is convenient to notice that

$$-i\bar{\psi}_+\partial_z\psi_+ + i\bar{\psi}_-\partial_z\psi_- = -i\bar{\chi}\partial_z\eta - i\bar{\eta}\partial_z\chi. \tag{34}$$

Eventually, we would like to find an equation for the surviving zero mode $\chi(x^\alpha, z)_0$ in the presence of a background scalar field which will be assumed to be real. For this purpose, let us write the surviving zero mode χ as

$$\chi_0\big(x^\alpha, z\big) = \chi\big(x^\alpha\big)\xi_{\chi,0}(z). \tag{35}$$

As we shall see, upon using Eq. (34) and subsequent interaction terms, one can derive an equation governing the behaviour of $\xi_{\chi,0}(z)$ along z which will eventually tell us whether or not one has a localized behaviour as in the five-dimensional case or an oscillatory one (pure phase). This will depend on the type of fermion bilinears which couple to the background scalar. Roughly speaking, if the coupling ends up to be of the form $i\bar{\eta}\chi h(z)$, for example, then $\xi_{\chi,0}(z)$ will have an exponentially-suppressed form similar to the five-dimensional case. If, however, it ends up looking like $\bar{\eta}\chi h(z)$, then $\xi_{\chi,0}(z)$ will have an oscillatory behaviour. This is so because of the way Eq. (34) looks.

We now look for the aforementioned fermion bilinears which are required to be Hermitian (because the background scalar field is assumed to be real) and Lorentz invariant.

Let us introduce a real scalar field which transforms under Z_2 as

$$\Phi\big(x^\alpha, z\big) \to -\Phi\big(x^\alpha, L_6 - z\big). \tag{36}$$

First, the most obvious, Hermitian and Lorentz-invariant bilinear is simply (remembering that Γ_0 is anti-Hermitian with our metric)

$$i\bar{\psi}\big(x^\alpha, z\big)\psi\big(x^\alpha, z\big). \tag{37}$$

Notice that (37), when expanded in terms of χ and η, are of the form $i\bar{\eta}\chi + \cdots$. This, when combined with Eq. (34), would give an exponentially-suppressed form for the zero mode if there exists such a Yukawa coupling. Can it couple to Φ? If the reflection Z_2 symmetry were the only symmetry around, it is straightforward to see that a coupling of the form $i\bar{\psi}(x^\alpha, z)\psi(x^\alpha, z)\Phi(x^\alpha, z)$ is an invariant. This, as we have mentioned above, would not be what we are looking for, namely, an oscillatory wave function. A mere mimicking of the five-dimensional case would not work. Below we propose a mechanism where the desired behaviour could arise.

Let us endow the scalar and fermion fields with an additional discrete symmetry which will be called the Q-symmetry and which works as follows. Let us divide the space inside the brane of thickness L_6 into two regions: 0 to $L_6/2$ (region I) and $L_6/2$ to L_6 (region II). Let us define the following transformations. Under Q,

$$\Phi\left(x^\alpha, z\right) \to -\Phi\left(x^\alpha, z\right). \tag{38}$$

Notice that (38) is not to be confused with (36) which is a reflection symmetry. We then notice the following fact: if z is inside region I then $L_6 - z$ will be inside region II and vice versa. For the fermion, we will impose the following Q-transformations: $\psi \to \psi$ for z in region I and $\psi \to -\psi$ for z in region II.

With the above Q-symmetry, one notices that a coupling of the form $\bar{\psi}(x^\alpha, z)\psi(x^\alpha, z) \times \Phi(x^\alpha, z)$ is forbidden for any point z inside the brane. However, a non-local interaction of the form $\bar{\psi}(x^\alpha, z)\psi(x^\alpha, L_6 - z)\Phi(x^\alpha, z)$ is allowed by the Q-symmetry. In particular, a *Hermitian* bilinear containing $\bar{\psi}(x^\alpha, z)\psi(x^\alpha, L_6 - z)$ of the form $\bar{\psi}(x^\alpha, z)\psi(x^\alpha, L_6 - z) - \bar{\psi}(x^\alpha, L_6 - z)\psi(x^\alpha, z)$ is allowed by this symmetry.

The way the Q-symmetry works seems to imply that the orbifold we used for the compactification should be $S_1/Z_2 \times Z_2'$ instead of a S_1/Z_2. The behaviour of the fields under the new Z_2' symmetry is, in fact, very similar to its behaviour under the initial one. To see this let us define $z' = z - L_6/2$ and:

$$\tilde{\psi}\left(x^\alpha, z'\right) = \psi\left(x^\alpha, L_6/2 + z'\right) = \psi\left(x^\alpha, z\right). \tag{39}$$

Again, from Eq. (18), we can see that the Lagrangian is invariant under the Z_2' symmetry:

$$\tilde{\psi}\left(x^\alpha, z'\right) \to \widetilde{\Psi}\left(x^\alpha, z'\right) = \Gamma_z \tilde{\psi}\left(x^\alpha, L_6 - z'\right). \tag{40}$$

We will impose the same boundary condition as for Z_2:

$$\tilde{\psi}\left(x^\alpha, z'\right) = \widetilde{\Psi}\left(x^\alpha, L_6 + z'\right). \tag{41}$$

Combining Eqs. (40) and (41) we get:

$$\begin{aligned} \tilde{\psi}_\pm\left(x^\alpha, -z'\right) &= \tilde{\psi}_\mp\left(x^\alpha, z'\right), \\ \tilde{\psi}_\pm\left(x^\alpha, L_6 - z'\right) &= \tilde{\psi}_\mp\left(x^\alpha, L_6 + z'\right) \end{aligned} \tag{42}$$

which, in terms of ψ and z, become:

$$\begin{aligned} \psi_\pm\left(x^\alpha, z\right) &= \psi_\mp\left(x^\alpha, L_6 - z\right), \\ \psi_\pm\left(x^\alpha, -z\right) &= \psi_\mp\left(x^\alpha, L_6 + z\right). \end{aligned} \tag{43}$$

Using this second parity we can find an explicit realization of the Q-symmetry as follows. First, we shall define the behaviour of the fermions under this symmetry in the region I as,

$$\psi'(z) = Q\psi(z) = \Gamma_7\psi(z) \tag{44}$$

now, Eq. (43) relates region I one and region II of the orbifold—as it should be since the physical space in a $S_1/Z_2 \times Z_2'$ goes from 0 to $L/2$—so in order for Q to be a symmetry of the Lagrangian the fermions have to satisfy,

$$Q\psi(L_6 - z) = \psi'(L_6 - z) = \Gamma_z\psi'(z) = \Gamma_z\Gamma_7\psi(z) = -\Gamma_7\psi(L_6 - z), \tag{45}$$

where in the second and last equalities we have used Eq. (43) which can also be written as $\psi(x^\alpha, z) = \Gamma_z\psi(x^\alpha, L_6 - z)$.

Notice that this realization of the Q-symmetry is only possible in an even number of space–time dimensions since it is only in this case that there exists a matrix which anticommutes with all of the gamma matrices of the algebra and which does not belong to the algebra.

With the above definitions, it is straightforward to see that the Yukawa coupling

$$\mathcal{L}_Y = \frac{f}{2}\Big(\bar\psi\big(x^\alpha, z\big)\psi\big(x^\alpha, L_6 - z\big) - \bar\psi\big(x^\alpha, L_6 - z\big)\psi\big(x^\alpha, z\big)\Big)\Phi\big(x^\alpha, z\big), \tag{46}$$

is invariant under all Z_2, Z_2' and Q symmetries where $Q\Phi(x^\alpha, z) = \Phi(x^\alpha, z)$. Furthermore, the action of the three parities forbids the presence of another non-local Hermitian term, $i(\bar\psi(z)\psi(L_6 - z) + \bar\psi(L_6 - z)\psi(z))\Phi(x^\alpha, z)$. In fact, Eq. (43) renders the above term to be identical to zero.

In terms of χ and η. Eq. (46) becomes

$$\begin{aligned}\mathcal{L}_{Y1} = \frac{f}{2}\Big\{&\Big(\bar\chi\big(x^\alpha, z\big)\eta\big(x^\alpha, L_6 - z\big) - \bar\chi\big(x^\alpha, L_6 - z\big)\eta\big(x^\alpha, z\big)\Big)\\ &- \Big(\bar\eta\big(x^\alpha, z\big)\chi\big(x^\alpha, L_6 - z\big) - \bar\eta\big(x^\alpha, L_6 - z\big)\chi\big(x^\alpha, z\big)\Big)\Big\}\Phi\big(x^\alpha, z\big).\end{aligned} \tag{47}$$

As before, the minimum energy solution for Φ is

$$\langle\Phi\rangle = h(z). \tag{48}$$

From (34) and (47), the equation of motion for the surviving zero mode $\xi_{\chi,0}(z)$ has the form:

$$-\partial_z\xi_{\chi,0}(z) + ifh(z)\xi_{\chi,0}(L_6 - z) = 0\,. \tag{49}$$

In order to solve Eq. (49), we shall use Eq. (43) that, in terms of χ and η leads to,

$$\xi_{\chi,0}(L_6 - z) = \xi_{\chi,0}(z)\,. \tag{50}$$

Because of the factor i in Eq. (49) $\xi_{\chi,0}(z)$ *will not* be localized along z.

The solution to (49) with the ansatz (50) is now given by

$$\xi_{\chi,0}(z) = \frac{1}{\sqrt{L}}e^{is(z)}, \tag{51}$$

where

$$s(z) = f\int\limits_0^z dz'\,h(z'). \tag{52}$$

Making the SHO approximation as used in the five-dimensional case (a) statement to be justified below, the properly normalized wave function for $\xi_{\chi,0}(z)$ would be

$$\xi_{\chi,0}(z) = \frac{1}{\sqrt{L_6}}e^{i\mu^2z^2}. \tag{53}$$

From the above solution for the zero mode in the 6th dimension, Eqs. (51), (53), we notice a marked difference with the 5-dimensional case: the zero mode wave function is now oscillating inside the thick brane, along the sixth dimension, while in the five-dimensional case, its counterpart has a localized form along the fifth dimension.

Let us assume there is a kink solution for Φ, i.e.,

$$h(z) = v\tanh(\mu z), \tag{54}$$

where $\mu = (\lambda/2)^{1/2}v$. With this solution (54) put into (52), the explicit expression for the non-vanishing zero mode is now

$$\xi_{\chi,0}(z) = \frac{1}{\sqrt{L_6}} e^{ifv\ln(\cosh(\mu z))/\mu}. \tag{55}$$

Just as we have done with the five-dimensional case, one could generalize the above discussion to include a "mass term" so that $fh(z) \to fh(z) - m$. As a result, one now has

$$\xi_{\chi,0}(z) = \frac{1}{\sqrt{L_6}} e^{i(fv\ln(\cosh(\mu z))/\mu - mz)}. \tag{56}$$

This more general expression (56) in fact determines the phase of the oscillation.

In the construction of the mass matrices in four dimensions, we will need overlaps of wave functions in the extra dimensions, as we have discussed above in regards with the fifth dimension. How the mass matrices look like in six dimensions is the topic which will be discussed next.

We end this section by presenting another type of Yukawa coupling which is used to actually localize fermions along the fifth dimension. The only difference with the previous section is that we now write it using the full six dimensions. With Γ_7 defined in Appendix A, the appropriate coupling is

$$S_{\text{Yuk2}} = \int d^6x\, f'\bar{\psi}\Gamma_7\Phi'\psi. \tag{57}$$

Defining $\tilde{\gamma}_5 = i\Gamma_y\Gamma_7$, one can see that Eq. (57) is invariant under $\psi(x^\mu, y, z) \to \pm\tilde{\gamma}_5\psi(x^\mu, L_5 - y, z)$ and $\Phi'(x^\mu, y, z) \to -\Phi'(x^\mu, L_5 - y, z)$ which finally gives $\psi_\pm(x, -y, z) = \pm\gamma_5\psi_\mp(x, y, z)$ and $\Phi'(x, -y, z) = -\Phi'(x, y, z)$. Also Eq. (57) is invariant under the Q-symmetry provided that $Q\,\Phi'(x^\mu, y, z) = -\Phi'(x^\mu, y, z)$. Notice that Eq. (57) can also be written as

$$S_{\text{Yuk2}} = \int d^6x\, f'\big(\bar{\chi}\Phi'\chi - \bar{\eta}\Phi'\eta\big). \tag{58}$$

Eq. (58) will reduce to the usual coupling in five dimensions. One last comments in order. Eq. (57) is also invariant under a simultaneous Z_2-transformation: $\psi(x^\alpha, z) \to \Gamma_z\psi(x^\alpha, L_6 - z)$, $\Phi'(x, y, z) \to \Phi'(x, y, L_6 - z)$, as well as under the Q-symmetry.

Before leaving this section, we would like to make a remark concerning Eq. (43). Basically, it is a "mapping" of region I into region II and vice versa, namely, $\psi(x^\alpha, z) = \Gamma_z\psi(x^\alpha, L_6 - z)$ or $\psi(x^\alpha, L_6 - z) = \Gamma_z\psi(x^\alpha, z)$. Now, let us remember that Eq. (43) is a consequence of our boundary conditions. When we substitute it into Eq. (46) so that one deals with the physical space which is now ranging from 0 to $L_6/2$, it acquires a Lorentz non-invariant form $\bar{\psi}(x^\alpha, z)\Gamma_z\psi(x^\alpha, z)$. What this says is that our boundary conditions break the six-dimensional Lorentz invariance down to a five-dimensional Lorentz invariance. Our original Lagrangian (46) is *Lorentz invariant* under the full six-dimensional Lorentz group and only when one goes to the physical space dictated by the boundary conditions, the six-dimensional Lorentz invariance is broken down to the five-dimensional one.

3.2. (Almost) pure phase mass matrices

We shall use the same notations as in Section 2.2. The action for the Yukawa interaction, in six dimensions, between the quarks and the SM Higgs field, is written as (the down sector is treated in exactly the same manner)

$$S_{\text{Yukawa}} = \int d^6x\, \kappa_U \sum_i Q_i^T C_6 H \sum_j U_j^c + \text{h.c.}, \tag{59}$$

where $C_6 = \Gamma_0\Gamma_2\Gamma_z$. We have, for the moment, omitted to write down other possible terms which are needed to determine the phases along the sixth dimension. This will be dealt with in the next section. We first begin with a "phenomenological" analysis.

The previous analysis led us to write a generic (zero-mode) fermion field as

$$\Psi(x, y, z) = \psi(x)\xi_5(y)\xi_6(z). \tag{60}$$

Before making use of Eq. (59) to construct the mass matrix, let us describe a possible "geography" of the fermions along the extra dimensions. The discussion of Section 2.2 pointed out the following features: the localization, along the fifth dimension y, of Q_i at one place and U_i^c at another place produces a democratic mass matrix as shown in Eq. (16). That is the "geography" along the fifth dimension that we would like to keep. Basically, left- and right-handed fields are localized by two domain walls at different locations. Why this should be so is beyond the scope of this paper. However, one important point that should be kept in mind is the fact that, in our model, there are only two locations (left and right) along the fifth dimension, regardless of the family index, for each quark sector (up or down). As mentioned above, this gives rise to the universal effective Yukawa couplings $g_{Y,u}$ and $g_{Y,d}$ which determine the overall mass strength for each sector. Let us recall that $g_{Y,u}$ and $g_{Y,d}$ are proportional to the overlap between left and right for the up and down sectors, respectively. Again, what splits $g_{Y,u}$ from $g_{Y,d}$ is beyond the scope of this paper. However, we will make some remarks concerning this issue at the end of the paper.

The next question concerns the locations of various domain walls along the sixth dimension. At the end of this section, we will present a simple example which shows how one can localize these domain walls. For the moment, we will simply parametrize these locations as shown in Eq. (56). We will assume that the domain walls which "fix" the phases for the three families are located at different positions along z. For the purpose of illustration, we will stay with this simple picture of family breaking in this manuscript. A more general case with phenomenological applications will be dealt with elsewhere. This will involve different profiles for different family kinks, etc.

We shall discuss below the implications of the cases when, for each family, Q and U^c are "in phase" and when they are slightly "out of phase". But, first, let us use Eq. (60) and

Eq. (59) to construct a general generic mass matrix for the up sector. The mass matrix for the down sector will be obtained in exactly the same manner.

In the following, the quantity L_6 which appears in various formulas is a generic symbol for the length of the physical space, which is L_6 itself for the orbifold S_1/Z_2 or $L_6/2$ for the orbifold $S_1/(Z_2 \times Z_2')$.

To begin, we will assume the following situation for the "geography" of family domain walls along the sixth dimension z. We will then discuss special cases of such a scenario. (As we have briefly mentioned above, this scenario is presented for the purpose of illustration and is not the most general case.) Let us define the following quantities which appear in Eq. (56):

$$f v_i/\mu_i \equiv a_i, \qquad m_{i;Q,U^c} \equiv m_{i,\mp}, \tag{61}$$

where $i = 1, 2, 3$ denotes the family index and where $\mu_i = (\lambda/2)^{1/2} v_i$. Notice that, in principle, the quartic coupling λ can depend on the family index i. This more general case, however, will be investigated elsewhere. From Eqs. (59), (60), one can write an effective Yukawa interaction in four dimensions and construct a mass matrix as we had done earlier. This construction is identical to the five-dimensional case, except that now the matrix elements will contain an extra factor which is the overlaps of $\xi_6(z)$'s. As usual, the mass matrix will be similar to Eq. (16) except that now, instead of the matrix elements being unity, one has

$$\mathcal{M} = g_{Y,u} \frac{v}{\sqrt{2}} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{62}$$

where

$$\begin{aligned} a_{jj} &= \int dz\, \xi^*_{6,j+} \xi_{6,j-} \\ &= \frac{1}{L_6} \int_0^{L_6} dz\, \exp\bigl(i(m_{j+} - m_{j-})z\bigr) \\ &= \bigl(\exp\bigl(i(m_{j+} - m_{j-})L_6\bigr) - 1\bigr) \big/ i(m_{j+} - m_{j-})L_6, \end{aligned} \tag{63a}$$

$$\begin{aligned} a_{ij} &= \int dz\, \xi^*_{6,i+} \xi_{6,j-} \\ &= \frac{1}{L_6} \int_0^{L_6} dz\, \exp\Bigl(i\bigl(a_j \ln\bigl(\cosh(\mu_j z)\bigr) - a_i \ln\bigl(\cosh(\mu_i z)\bigr) + (m_{i+} - m_{j-})z\bigr)\Bigr). \end{aligned} \tag{63b}$$

Notice that L_6 here is a generic symbol for the length of the physical space as we have mentioned above.

The above equations (62), (63a), (63b) refer to the case where domain walls, which "determine" the phases of the fermions, are "located" at different places. We will specialize below to a few interesting possibilities. However, some important remarks can already be made. We ask the following question: under what conditions will the mass matrix be Hermitian or non-Hermitian?

3.2.1. Hermitian and non-Hermitian mass matrices

We now present two different scenarios.

(a) The parameters $m_{i\pm}$ which determine the "locations" of the domain walls possess interesting features. The first observation one can make is as follows. If the domain walls which "localize" the phases of Q and U^c (left and right), for *each* family, are located at the same place along z, i.e.,

$$m_{i+} = m_{i-} \tag{64}$$

one obtains the following results

$$a_{jj} = 1, \tag{65a}$$

$$a_{ji} = a_{ij}^*. \tag{65b}$$

The mass matrix $\mathcal{M}$ is *Hermitian*! The hermiticity of the mass matrix is a *consequence* of the "collapse" of left and right (or Q and U^c), for each family, into the "same position" along the sixth dimension. Two remarks can be made concerning a Hermitian matrix. First, its determinant is *real*. This means that $\arg(\det\mathcal{M}) = 0$. The possible connection of this statement with the strong CP problem (see, e.g., a review by [8]) will be explored further at the end of the paper.

Let us first see if the Hermitian matrix above is of a pure phase form.

The discussion which follows will deal with issues which are also relevant to the non-Hermitian case.

Let us look at

$$a_{ij} = \frac{1}{L_6}\int_0^{L_6} dz\, \exp\Bigl(i\bigl(a_j \ln\bigl(\cosh(\mu_j z)\bigr) - a_i \ln\bigl(\cosh(\mu_i z)\bigr) + (m_i - m_j)z\bigr)\Bigr). \tag{66}$$

Under what conditions would a_{ij}'s look like pure phases, namely, of the form $e^{i\theta}$, or an almost pure phase of the form $(1-\rho)e^{i\theta}$ with $\rho \ll 1$? To answer this question, let us make a little detour to the meaning of wave function overlaps, thickness of domain walls and size of the extra dimensions.

We have seen how one can localize fermions along the fifth dimension (y) by having domain walls of sizes $1/\mu \ll L_5$. The effective strengths of various interactions are determined by the overlaps of the wave functions along y. For this reason, it is preferable to have the thickness of the domain walls small enough, i.e., $1/\mu \ll L_5$, so one can "fit" several fermions along y in such a way as to obtain desirable effects such as "slow" (or no) proton decay, possible mass hierarchies between different fermion sectors (quarks, leptons), etc. As we move on to the sixth dimension, it is not obvious that such a picture is still necessary. In fact, at least as far as the pure phase mass matrix is concerned, the thickness of these domain walls can be as large as the size of the compactified dimension itself, as we shall see below.

Let us, for the time being, assume that all domain wall thicknesses (along z) are of the size of the compact dimension, i.e., $1/\mu_i \sim O(L_6)$. In this situation, one can use the SHO approximation and carry out the integration of Eq. (66), namely,

$$a_{ij} = \frac{1}{L_6} \int_0^{L_6} dz \, \exp\bigl(-i\bigl(\Delta\mu_{ij}^2 z^2 - \Delta m_{ij} z\bigr)\bigr), \tag{67}$$

where

$$\Delta\mu_{ij}^2 \equiv (1/2)\bigl(a_i \mu_i^2 - a_j \mu_j^2\bigr), \tag{68a}$$

$$\Delta m_{ij} \equiv m_i - m_j. \tag{68b}$$

The integration can be explicitly carried out. One obtains

$$a_{ij} = \frac{\sqrt{\pi}}{2} \frac{\operatorname{erf}\bigl(\frac{i(2\Delta\mu_{ij}^2 L_6 - \Delta m_{ij})}{2\sqrt{i\Delta\mu_{ij}^2}}\bigr) + \operatorname{erf}\bigl(\frac{i\Delta m_{ij}}{2\sqrt{i\Delta\mu_{ij}^2}}\bigr)}{\sqrt{i\Delta\mu_{ij}^2}\, L_6} \exp\biggl(i\frac{(\Delta m_{ij})^2}{4\Delta\mu_{ij}^2}\biggr). \tag{69}$$

In a phenomenological application of Eq. (69), one can use it without making any approximation. However, in order to see if it has a more familiar pure phase form or not, we will make an expansion of (69).

Let us define

$$\sqrt{\Delta\mu_{ij}^2}\, L_6 \equiv x_{ij}, \tag{70}$$

$$\Delta m_{ij}\, L_6 \equiv y_{ij}. \tag{71}$$

For $x_{ij},\ y_{ij} < 1$, one can expand (69) giving

$$a_{ij} = \biggl\{1 - \frac{2}{45} x_{ij}^4 - \frac{1}{24} y_{ij}^2 + \frac{1}{12} x_{ij}^2 y_{ij}\biggr\} \exp\biggl\{i\biggl(\frac{y_{ij}}{2} - \frac{x_{ij}^2}{3}\biggr)\biggr\}, \tag{72}$$

where we have neglected terms of $O(x_{ij}^8, y^4)$ or less in the modulus and terms of $O(x_{ij}^6, y^4)$ in the phase. Notice that for a_{ji}, one has $x_{ji}^2 = -x_{ij}^2$ and $y_{ji} = -y_{ij}$, and hence $a_{ji} = a_{ij}^*$ as they should. In this form one can see that the Hermitian mass matrix is *almost* of the pure phase form. This would have been the case if one could neglect terms containing x_{ij} and y_{ij} inside the coefficient multiplying the exponential. However, we will not neglect those terms, leaving the possibility of a small deviation [9] from a pure phase mass matrix.

Notice that when the domain walls are all located at the same point, i.e., $\Delta m_{ij} = 0$, and when they have the same thickness (or $\mu_i = \mu_j$), $\forall i, j$, one recovers the DMM form, namely, $a_{ij} = 1$, as one can see from Eqs. (66), (67), (72). In addition, we notice that one can also obtain the almost-pure phase Hermitian mass matrix when either $\Delta m_{ij} \neq 0$ or $\Delta\mu_{ij}^2 \neq 0$, but not necessarily both, as can easily be seen.

(b) As we have seen above, within the framework of Eqs. (63a), (63b), the mass matrix can be purely Hermitian provided the condition $m_{i+} = m_{i-}$ is fulfilled. What would happen if $m_{i+} \neq m_{i-}$? To study this question, let us refer back to Eqs. (63a), (63b), (72) and let

$$m_{i+} - m_{i-} = \epsilon_i. \tag{73}$$

Also for convenience, let us define

$$\delta_i = \epsilon_i L_6. \tag{74}$$

With the above definitions, the diagonal matrix elements which are no longer unity, can be written as

$$a_{ii} = \exp(i\delta_i/2)\frac{\sin(\delta_i/2)}{(\delta_i/2)}. \tag{75}$$

The off-diagonal elements are similar to Eq. (72), except that now one has the following replacement $y_{ij} \to y_{i+,j-} = (m_{i+} - m_{j-})L_6$. It is convenient to remove the phases from the diagonal elements by absorbing the phases into $\xi_{6,i+}$, namely, $\xi_{6,i+} = \exp(i\delta_i/2)\,\xi'_{6,i+}$. From the definitions of a_{ii} and a_{ij}, one now has

$$a_{ii} = \frac{\sin(\delta_i/2)}{(\delta_i/2)}, \tag{76}$$

$$\begin{aligned} a_{ij} = {}& \left\{1 - \frac{2}{45}x_{ij}^4 - \frac{1}{24}y_{i-,j-}^2 + \frac{1}{12}x_{ij}^2 y_{i-,j-} - \frac{1}{12}y_{i-,j-}\delta_i - \frac{1}{24}\delta_i^2 + \frac{1}{12}x_{ij}^2\delta_i\right\} \\ & \times \exp\left\{i\left(\frac{y_{i-,j-}}{2} - \frac{x_{ij}^2}{3}\right)\right\}, \end{aligned} \tag{77}$$

where

$$y_{i-,j-} = (m_{i-} - m_{j-})L_6. \tag{78}$$

Notice that $y_{i-,j-} = -y_{j-,i-}$. In Eq. (77), we have made use of the above phase redefinition and of (73), (74). The mass matrix described by the above elements is *not* Hermitian for the following reason. The modulus of a_{ji} will have a term $-\frac{1}{12}y_{j-,i-}\delta_j - \frac{1}{24}\delta_j^2 + \frac{1}{12}x_{ji}^2\delta_j = \frac{1}{12}y_{i-,j-}\delta_j - \frac{1}{24}\delta_j^2 - \frac{1}{12}x_{ij}^2\delta_j$. It can easily be seen that $|a_{ij}| \neq |a_{ji}|$ unless $\delta_j = -\delta_i$ which cannot be satisfied for all j. Despite the fact that the phase of a_{ji} is the negative of that of a_{ij}, the difference in moduli implies that, in general, $a_{ji} \neq a_{ij}^*$, and hence the *non-hermiticity* of the matrix. It can be approximately Hermitian if one can neglect the terms containing δ_i in (77).

Notice that, even for the special case where all "left-handed" family domain walls are "located" at one point along z, i.e., $m_{i-} = m_{j-} = m_-$, so that $y_{i-,j-} = 0$, and all "right-handed" family domain walls at another place, i.e., $\delta_i = \delta$, the non-hermiticity still appears in the difference in moduli between a_{ji} and a_{ij} because of the presence of δ.

From the above discussion, one can see that one recovers the Hermitian matrix in the limit $\delta_i \to 0$.

In summary, we have shown that, in general, the deviation from hermiticity in our framework comes from the splitting between "left" and "right", namely, $m_{i+} \neq m_{i-}$.

The above analysis can be carried over to the down sector in exactly the same manner. There are, however, two interesting remarks that can be made. First, although the mass matrix for the down sector is now characterized by a universal strength $g_{Y,d}$ which is in general different from $g_{Y,u}$, the matrix itself can be identical to the one for the up sector if we consider scenario (a). The reason is that scenario (a) is one in which the domain walls for Q and D^c, for each family, are "located" at the same place along the sixth dimension, which is exactly the same as for the up sector. Therefore, the matrix elements (without the universal strength) are *the same*. In consequence, the diagonalization matrices are *the same*, i.e., $V_U \equiv V_D$. Hence, $V_{\mathrm{CKM}} = V_U^\dagger V_D = 1$, a mere unit matrix. In other words, the mass matrices for the up and down sectors cannot be *both* Hermitian. To obtain a non-trivial CKM matrix, at least one of the two matrices has to be non-Hermitian in this particular scenario.

The above (almost) pure phase mass matrix as obtained from six dimensions is what we have set out to derive. From it, we have learned a few things.

(a) In general, the almost pure phase form of the mass matrix can be easily seen if the thickness of various domain walls along the sixth dimension is of the order of the compactified sixth dimension. (There is no reason why, in principle, the thickness of the domain walls should be much smaller than the compactified dimension, in contrast with the five-dimensional case.)

(b) When the domain walls "fixing" the phases for Q and U^c, *for each family*, are located at the same place, ($m_{u,i+} = m_{u,i-}$), the mass matrix is purely Hermitian. As we have seen above, another possibility is when the domain walls "fixing" the phases for Q are at one location and those which are responsible for "fixing" the phases of U^c are at another location, in which case the mass matrix is also Hermitian. If one considers these cases to be a "tree-level" situation (a) statement to be further clarified below, the fact that $\arg(\det \mathcal{M}) = 0$ makes this scenario an interesting "candidate" for a solution to the strong CP problem.

(c) The mass matrix becomes *non-Hermitian* when $m_{u,i+} \neq m_{u,i-}$. We will briefly discuss below the possibility that $m_{u,i+} \neq m_{u,i-}$ is due to "radiative corrections" of the case $m_{u,i+} = m_{u,i-}$.

The mass matrix for the down sector is obtained in a similar way. The main difference between the two sectors is the "universal" strength which appears in front of the matrix: $g_{Y,u}\frac{v_u}{\sqrt{2}}$ for the up sector and $g_{Y,d}\frac{v_d}{\sqrt{2}}$ for the down sector. The other difference in the case of a non-Hermitian matrix (scenario (b)) is the splitting between "left" and "right" for each family, which does not have to be the same for the two sectors.

Notice that, in order to be more general, we allow the possibility of two different mass scales: v_u and v_d. If there were only *one* SM Higgs field then $v_u = v_d = v$. In this case, the disparity between the mass scales of the up and down sectors would come from the difference between $g_{Y,u}$ and $g_{Y,d}$, which, in turns, could come from the differences between wave function overlaps, along the fifth dimension, of the two sectors (modulo differences in the fundamental Yukawa couplings). To keep our discussions as general as possible, we also allow for the possibility that two SM Higgs fields exist.

It is beyond the scope of this paper to discuss in detail the phenomenology of our model. It will be carried out elsewhere.

3.2.2. Some remarks on localization of family domain walls along the sixth dimension

In this section, we will briefly discuss one way to localize the various domain walls responsible for "fixing" the phases of fermions along the sixth dimension. There are probably several mechanisms to achieve this. We will present one of such mechanisms, from the point of view of effective field theory.

For simplicity, we shall assume in this section that $a_i = a = f/\sqrt{\lambda/2}$. This simple assumption basically refers to couplings between fermions and background scalar fields which are invariant under the family symmetry.

First, let us list the parameters that we need to construct an almost pure phase mass matrix. From Section 3.2.1, we learned that we need: μ_i with $i = 1, 2, 3$ which control the thicknesses of the domain walls and $m_{i\pm}$ which control the locations of the domain walls. We also learned that, one can obtain a Hermitian mass matrix when $m_{i+} = m_{i-}$ and a non-Hermitian matrix when $m_{i+} \neq m_{i-}$. It turns out to be a highly non-trivial task to find a mechanism which can "explain" the origin of these parameters. In some sense, it might even be overly ambitious to make such a claim. We will, however, make an attempt to, at least, hint at one possible scenario.

In Section 3.2.1, we were basically doing the "geography" of family domain walls along the sixth dimension. To construct a scenario for the "geographical points" (the various m's), let us recall that the family symmetry of our model is $S_3^Q \otimes S_3^{U^c}$. The background scalar fields which couple to Q or U^c will appear in terms such as $\overline{Q}\Phi Q$, $\overline{U}^c \Phi U^c$. We will, therefore, need two of such background fields in order to write down invariant Yukawa couplings: Φ_Q and Φ_{U^c}. These background fields, Φ_Q and Φ_{U^c}, will be represented by 3×3 matrices. Some of the details concerning the potential for these scalars are given in Appendix B. Here, we will just quote the results. The discussion below refers to the up sector. As we have seen earlier, the down sector can be treated in exactly the same manner.

We will concentrate on scenario (a) of Section 3.2.1 for the purpose of illustration. We will assume the following Yukawa interactions:

$$\mathcal{L}_Y = f\,\overline{Q}\Phi_Q Q + f\,\overline{U}^c \Phi_{U^c} U^c + \text{h.c.}, \tag{79}$$

where, for simplicity, we have put the two Yukawa couplings to be equal. (A more general case can be accommodated straightforwardly.) The minimization of the potential gives, at *tree level*,

$$\langle \Phi_Q \rangle = \begin{pmatrix} h_1(z) & 0 & 0 \\ 0 & h_2(z) & 0 \\ 0 & 0 & h_3(z) \end{pmatrix}. \tag{80}$$

One could assume that, at some deeper level and because of the family symmetry, the two background fields behave in exactly the same manner, i.e., having similar parameters, and, in consequence, one has

$$\langle \Phi_{U^c} \rangle = \begin{pmatrix} h_1(z) & 0 & 0 \\ 0 & h_2(z) & 0 \\ 0 & 0 & h_3(z) \end{pmatrix}. \tag{81}$$

These VEVs will be shifted by radiative corrections. It is beyond the scope of this paper to examine this problem and we will simply parametrize these shifts by

$$h_i(z) \to h_i(z) + \delta h_i, \tag{82}$$

where the shifts are assumed to be independent of z and are also assumed to be much smaller than v_i (or μ_i).

Combining Eq. (82) with Eq. (79), one can make the following identification

$$m_{i-} = m_{i+} = f\,\delta h_i. \tag{83}$$

This is the case when one would obtain a Hermitian mass matrix of scenario (a) of Section 3.2.1! It goes without saying that there are two assumptions which have been made. First, we have assumed the equality of the Yukawa couplings in Eq. (79). Second, we have assumed that the behaviour of the two background scalar fields are identical. These assumptions might come from some deeper symmetry between Q and U^c (or D^c). This is very similar to the notion of left–right symmetry that one encounters in four-dimensional model building. In consequence, the hermiticity of the mass matrix that we obtained by "phenomenologically" putting $m_{i-} = m_{i+}$ might be justified by some form of left–right symmetry.

In addition (82), one should also take into account vertex corrections which will be different for Q and U^c (they have different gauge interactions, for example). Let us parametrize those shifts by

$$\tilde{f}_Q = f + \delta f_Q, \tag{84a}$$

$$\tilde{f}_U = f + \delta f_{U^c}, \tag{84b}$$

where the notations are self-explanatory. We will assume that $\delta f_{Q,U^c} \ll f$. Naturally, $\tilde{f}_Q \neq \tilde{f}_U$.

From the above equations, one can make the following identifications:

$$m_{i-} = \tilde{f}_Q \delta h_i, \tag{85a}$$

$$m_{i+} = \tilde{f}_U \delta h_i. \tag{85b}$$

Since one expects $\delta f_Q \neq \delta f_{U^c}$ and, in consequence, $\tilde{f}_Q \neq \tilde{f}_U$, one would expect, in general, $m_{i+} \neq m_{i-}$ which is a condition for the appearance of a non-Hermitian matrix. However, an approximate Hermitian matrix could arise if the radiative corrections and, in particular, the difference in the radiative corrections are small. One can see that, as we turn off whatever interactions (gauge, etc.) which contribute to the vertex corrections $\delta f_{Q,U^c}$, one recovers the Hermitian case, namely, $m_{i-} = m_{i+}$.

Pursuing the same idea, one can also assume that D^c's have a similar coupling of the form $f\,\overline{D}^c \Phi_{D^c} D^c$. Assuming that $\langle \Phi_{D^c} \rangle$ has a similar form to Eqs. (80), (81), one can now see that, in the absence of vertex corrections, one obtains $m_{i-} = m_{i,u,+} = m_{i,d,+}$, which is just scenario (a) discussed above. Since D^c and U^c have different quantum numbers, one expects that their vertex corrections will be different from each other. In consequence, one will obtain mass matrices of the form (62) with coefficients of the form (63a), (63b).

In the scenario just outlined above, one can make interesting connections with the strong CP problem. In the absence of vertex corrections, the mass matrix is *Hermitian* and hence $\arg(\det \mathcal{M}) = 0$, a possible solution to the strong CP [8] problem? (One could assume CP to be a symmetry of the Lagrangian so that $\theta_{\mathrm{QCD}} = 0$.) As mentioned above, this hermiticity might come from some left–right symmetry ($Q \leftrightarrow U^c, D^c$) which gives $m_{i-} = m_{i+}$ at "tree level". It could be quite provocative to see if there are connections, if any, with previous solutions to the strong CP problem which made use of the quintessential left–right symmetry [13].

Turning on the vertex corrections, the pure phase mass matrix becomes non-Hermitian and, as a consequence, one would obtain a non-zero contribution to the strong CP parameter $\bar{\theta}$. If this were truly a plausible scenario for the strong CP problem, the resultant $\bar{\theta}$ should obey the upper bound of $\sim 10^{-9}$. However, it is beyond the scope of this paper to analyze its magnitude. We will come back to this issue in a subsequent paper. Our future studies will focus on the following two questions. Will the "radiative corrections" be small enough so as to account for both the phenomenological constraints on the mass matrices *and* the magnitude of $\bar{\theta}$? If those phenomenological constraints on the mass matrices require a "large" radiative correction, is there a "natural" mechanism to make $\bar{\theta}$ small enough?

4. Conclusion

In this paper, we have studied the problem of fermion mass hierarchy from the point of view of large extra dimensions. To this end, we have added two extra compact spatial dimensions. In particular, we have shown how one can construct a particular kind of mass matrices which is very successful in fitting the pattern of quark masses and mixing angles: the pure phase mass matrix. This matrix is characterized by a universal Yukawa strength appearing in front of a matrix whose elements are of the form $\exp(i\theta_{ij})$. In our construction, the universal Yukawa strength arises from the overlap of the wave functions of the left-handed quarks (denoted by Q) and the right-handed quarks (denoted by U^c and D^c) along the fifth spatial dimension (y). Along y, all left-handed families are localized at one place and all right-handed families at another place, with the localization carried out by domain walls whose thicknesses are assumed to be much smaller than the radius of compactification of y. We then proceed to show that the almost pure phase mass matrix arise from the overlap of wave functions between different families and also between left-handed and right-handed quarks, along the sixth dimension z. Along z, the "phase determination" is carried out by domain walls whose thicknesses are assumed to be of the size of the radius of compactification of z.

We would like to stress again that the results obtained in this paper depend purely on the "geography" of the domain walls along the extra dimensions.

The almost-pure phase mass matrices obtained in six dimensions have some interesting properties, according to the "locations" of the family domain walls, which fix the phases, along the sixth dimension. In one case (scenario (a)) which is dubbed "tree level" in this paper, the domain walls for Q and U^c or D^c are "located" at the same place along the sixth dimension z, for each family. The mass matrices thus obtained are purely *Hermitian*. In addition, apart from a different universal Yukawa strength, the matrices of the up and down sectors are identical, giving rise to a situation in which the CKM matrix is simply a unit matrix. We then considered a scenario in which either the domain wall for U^c or D^c, or both, is split from that for Q. As we have shown in Section 3.2.1, this would imply that the mass matrix of at least one of the two sectors is non-Hermitian, and the two matrices will be different from each other, implying a non-trivial CKM matrix.

One should also keep in mind the possibility that the mass matrices of both sectors are Hermitian but not identical. In this type of scenario, one would get a non-trivial

CKM matrix as well as a correct spectrum. This possibility is mentioned at the end of Section 3.2.1. In order to be able to build a model for the "locations" of various quarks along the extra spatial dimensions, a full phenomenological analysis of various possibilities should be carried out to serve as a guidance.

These two cases of Hermitian and non-Hermitian mass matrices might have important connections to the strong CP problem as we have briefly discussed above. This interesting issue will be further investigated in a future paper.

Finally, a number of interesting issues such as the Kaluza–Klein modes, the extension to the lepton sector, and others will be dealt with in future publications.

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Appendix A

In this appendix we are going to present a brief review of the spinorial representations of the orthogonal group, $O(D)$, in higher dimensions ($D > 4$). We are going to follow closely the treatment done by Weinberg in his book [14], with a slightly different notation. Notice that the first paper on this subject was written by Mohapatra and Sakita [15].

The starting point is a set of matrices, which spawn the Clifford algebra, with the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. In addition, our attention will be fixed on spaces of even dimensionality ($D = 2n$) and, subsequently, we will extend it to odd dimension spaces.

Using the anticommutation relations of the γ matrices, we can define n fermionic harmonic oscillators as $a_i^+ = \frac{1}{2}(-\gamma_{2i} + i\gamma_{2i+1})$ with $i = 0, \ldots, n-1$, that are independent and, therefore, the set of basis vectors of the representation space has 2^n elements which can be written as:

$$|s_1 s_2 \ldots s_n\rangle = a_1^{+s_1} a_2^{+s_2} \cdots a_n^{+s_n} |0\rangle \tag{A.1}$$

being $|0\rangle$ a vacuum annihilated by all destruction operators a_i. In this basis the matrices a_i take the form:

$$a_i = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I}_{2\times 2} \otimes \cdots \otimes \mathbb{I}_{2\times 2} \tag{A.2}$$

the -1's are due to the fact that a_i^+ and a_j^+ anticommute. Finally the γ matrices can be easily obtained and they read as:

$$\begin{aligned} \gamma_{2i} &= -\sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{I}_{2\times 2} \otimes \cdots \otimes \mathbb{I}_{2\times 2}, \\ \gamma_{2i+1} &= \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{I}_{2\times 2} \otimes \cdots \otimes \mathbb{I}_{2\times 2}, \end{aligned} \tag{A.3}$$

where σ_i's are the Pauli matrices. Note that this representation does not give the usual representation in four dimensions but we can relate both using the following unitary transformation:

$$U = \frac{1}{(\sqrt{2})^n}(\sigma_3 + \sigma_2) \otimes \cdots \otimes (\sigma_3 + \sigma_2) \tag{A.4}$$

so the "usual" representation is

$$\begin{aligned} \gamma_{2i} &= \sigma_2 \otimes \cdots \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbb{I}_{2\times 2} \otimes \cdots \otimes \mathbb{I}_{2\times 2}, \\ \gamma_{2i+1} &= \sigma_2 \otimes \cdots \otimes \sigma_2 \otimes \sigma_3 \otimes \mathbb{I}_{2\times 2} \otimes \cdots \otimes \mathbb{I}_{2\times 2}. \end{aligned} \tag{A.5}$$

Since,

$$\gamma_{2i}\gamma_{2i+1} = i\mathbb{I}_{2\times 2} \otimes \cdots \otimes \mathbb{I}_{2\times 2} \otimes \sigma_2 \otimes \mathbb{I}_{2\times 2} \otimes \cdots \otimes \mathbb{I}_{2\times 2} \tag{A.6}$$

the product of the $2n$ γ matrices is:

$$\prod_{i=0}^{2n-1} \gamma_i = i^n \sigma_2 \otimes \cdots \otimes \sigma_2 = \eta\gamma_{2n}. \tag{A.7}$$

Where η is a phase such that $\gamma_{2n}\gamma_{2n}$ is the identity. Note that we are labeling the gamma matrix equivalent to γ_5 with $2n$ instead $2n+1$. This difference comes from our choice for the labeling starting from 0 instead of 1. This new matrix anticommutes with all γ's and therefore it implies that all spinorial representations of $O(2n)$ are reducible.

Let us find now the spinorial representations of the orthogonal groups with odd dimensionality, $D = 2n+1$; this is much more simpler once we have the representation for $O(2n)$ since we just have to take this representation and add the γ_{2n} matrix. In this case the representation is irreducible because we cannot find any independent matrix that anticommutes with all the gamma matrices.

The transition of the $O(D)$ representations to $O(D-1,1)$ representations is done through a wick rotation.

To finalize we will explicitly write the gamma matrices for $O(5,1)$.

- Six dimensions (with metric $(-+++++)$):

$$\Gamma_0 = i\sigma_2 \otimes \sigma_1 \otimes \mathbb{I}_{2\times 2} = \begin{pmatrix} 0_{4\times 4} & -\gamma_0 \\ \gamma_0 & 0_{4\times 4} \end{pmatrix},$$

$$\Gamma_1 = \sigma_2 \otimes \sigma_2 \otimes \sigma_1 = \begin{pmatrix} 0_{4\times 4} & -\gamma_1 \\ \gamma_1 & 0_{4\times 4} \end{pmatrix},$$

$$\Gamma_2 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0_{4\times 4} & -\gamma_2 \\ \gamma_2 & 0_{4\times 4} \end{pmatrix},$$

$$\Gamma_3 = \sigma_2 \otimes \sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0_{4\times 4} & -\gamma_3 \\ \gamma_3 & 0_{4\times 4} \end{pmatrix},$$

$$\Gamma_y = \sigma_2 \otimes \sigma_3 \otimes \mathbb{I}_{2\times 2} = \begin{pmatrix} 0_{4\times 4} & -i\gamma_5 \\ i\gamma_5 & 0_{4\times 4} \end{pmatrix},$$

$$\Gamma_z = \sigma_1 \otimes \mathbb{I}_{2\times 2} \otimes \mathbb{I}_{2\times 2} = \begin{pmatrix} 0_{4\times 4} & \mathbb{I}_{4\times 4} \\ \mathbb{I}_{4\times 4} & 0_{4\times 4} \end{pmatrix},$$
$$\Gamma_7 = \sigma_3 \otimes \mathbb{I}_{2\times 2} \otimes \mathbb{I}_{2\times 2} = \begin{pmatrix} \mathbb{I}_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 4} & -\mathbb{I}_{4\times 4} \end{pmatrix}. \tag{A.8}$$

Notice that, in the above equations, γ^μ ($\mu = 0, 1, 2, 3$) and γ_5 are simply defined *here* as $\gamma_0 = \sigma_1 \otimes \mathbb{I}_{2\times 2}$, $\gamma_i = i\sigma_2 \otimes \sigma_i$, $\gamma_5 = \sigma_3 \otimes \mathbb{I}_{2\times 2}$. These definitions just happen to coincide with the 4-dimensional ones with a metric $(+---)$. This is simply a compact way of writing the 6-dimensional Γ's. There is *no change* in metric. To see how the 6-dimensional metric $(-+++++)$ reduces to a 4-dimensional metric $(-+++)$ when the two extra spatial dimensions are compactified, one rewrites γ^μ in terms of the gamma matrices which correspond to the metric $(-+++)$, namely, $\tilde{\gamma}^\mu = i\gamma^\mu$. In this way, the kinetic terms will be preceded with a plus sign when they are reexpressed in terms of $\tilde{\gamma}^\mu$. Of course, γ_5 remains unchanged.

Appendix B

In this article, we had been discussing models in which fermions are localized at one place or another along the extra dimensions and inside fat branes with the same or different widths and how these settings could affect the phenomenology of the 4D models. However, we did not provide any model that explains these different settings; this will be addressed in this appendix.

As an example, we are going to study the possibility that the background scalar field is a composite of fields that transforms under a three-dimensional representation of the family group; therefore, this background scalar field Φ takes the form:

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix} \otimes \big(\phi_1(x)\phi_2(x)\phi_3(x)\big)^+, \tag{B.1}$$

where the ϕ_i's are the "fundamental fields" from which the background scalar field is composed.

The first of the models we are going to propose consists on a ϕ^4 potential, without cubic terms, for 2 composite fields of the form (B.1),

$$\begin{aligned}
&V(\Phi_1, \Phi_2) \\
&\quad = \frac{m_1^2}{2}\operatorname{Tr}\big[\Phi_1\Phi_1^+\big] + \frac{m_2^2}{2}\operatorname{Tr}\big[\Phi_2\Phi_2^+\big] + \frac{m_3}{2}\big(\operatorname{Tr}\big[\Phi_1\Phi_2^+\big] + \text{h.c.}\big) \\
&\qquad + \frac{\lambda_1}{4}\operatorname{Tr}\big[\big(\Phi_1\Phi_1^+\big)^2\big] + \frac{\lambda_2}{4}\operatorname{Tr}\big[\big(\Phi_2\Phi_2^+\big)^2\big] + \frac{\lambda_3}{2}\operatorname{Tr}\big[\big(\Phi_1\Phi_1^+\big)\big(\Phi_2\Phi_2^+\big)\big] \\
&\qquad + \frac{\lambda_4}{4}\big(\operatorname{Tr}\big[\Phi_1\Phi_2^+\Phi_1\Phi_2^+\big] + \text{h.c.}\big) \\
&\quad = \frac{m_1^2}{2}|\Phi_1|^2 + \frac{m_2^2}{2}|\Phi_2|^2 + m_3|\Phi_1||\Phi_2|\cos^2\alpha
\end{aligned}$$

$$+\frac{\lambda_1}{4}|\Phi_1|^2+\frac{\lambda_2}{4}|\Phi_2|^2+\frac{\lambda_3}{2}|\Phi_1|^2|\Phi_2|^2\cos^2\alpha$$
$$+\frac{\lambda_4}{2}|\Phi_1|^2|\Phi_2|^2\cos^4\alpha, \tag{B.2}$$

where $m_{1,2}^2$ are negative coefficients. We also made use of the following definitions,

$$|\Phi|^2=\mathrm{Tr}\big[\Phi\Phi^+\big], \tag{B.3}$$

$$\cos^2\alpha=\mathrm{Re}\left(\frac{\mathrm{Tr}[\Phi_1\Phi_2^+]}{|\Phi_1||\Phi_2|}\right). \tag{B.4}$$

If $2m_3 > -(\lambda_3+\lambda_4\cos^2\alpha)|\Phi_1||\Phi_2|$ holds then the minimum of the potential occurs when both fields take expectation values along orthogonal directions, that is, when $\mathrm{Tr}[\Phi_1\Phi_2^+]=0$. In consequence, if we suppose that the expectation values for $|\Phi_1|$ and $|\Phi_2|$ are u and v, respectively, the potential we have to minimize is:

$$V(u,v)=\frac{m_1^2}{2}u^2+\frac{m_2^2}{2}v^2+\frac{\lambda_1}{4}u^4+\frac{\lambda_2}{4}v^4 \tag{B.5}$$

and its minimum is located at:

$$u=\sqrt{-\frac{m_1^2}{\lambda_1}},\qquad v=\sqrt{-\frac{m_2^2}{\lambda_2}}, \tag{B.6}$$

therefore we can suppose that,

$$\langle\Phi_1\rangle=\begin{pmatrix}u&0&0\\0&0&0\\0&0&0\end{pmatrix}, \tag{B.7}$$

$$\langle\Phi_2\rangle=\begin{pmatrix}0&0&0\\0&v&0\\0&0&0\end{pmatrix}. \tag{B.8}$$

This means that this model will localize the components of the family multiplet at different positions along the sixth dimension, namely, 0, u and v.

We can extend this model to localize all the components of the family multiplet inside the orbifold, with the background scalar field in (46) having all three eigenvalues different from zero. This can be done if we add another background field that can be a singlet, or a composite like the ones used. In the former case the three components will be shifted by the same amount, s, ending in positions $s+u$, $s+v$ and s; in the later case the fermions will be located at u, v and z (the expectation value for the third field).

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