ABSTRACT

The propagation of bosonic strings in background massive fields is considered. We present a prescription to compute the Weyl-anomaly coefficients of the sigma-model effective action in the presence of renormalizable as well as non-renormalizable operators. The field equations corresponding to the first mass levels of the open string are obtained by imposing conformal invariance on this effective action \textit{i.e.}, by imposing the requirement that the Weyl-anomaly coefficients vanish. The study reveals that the anomalous part of the effective action of the sigma model with operators of arbitrary dimension has a very simple structure.
1. Introduction

The analysis of string propagation in general background fields is of interest for two main reasons. From a phenomenological point of view, since it provides field equations, it may be useful to deal with compactifications of extra spacetime dimensions. From a formal point of view, it may give some insight in understanding the role of gauge invariances in string field theory. Work on this problem is based on the so-called sigma-model approach [1,2,3] which consists of a non-linear version of the Polyakov [4] path integral. The basic statement describing this approach is that the conditions of conformal invariance of the nonlinear sigma model imply field equations for the string modes. This statement has shown to be correct in the variety of situations in which it has been tested. These situations correspond to the massless modes of the closed [1,2,3,5] and open [6] strings at tree-string level and different orders of sigma-model loops, as well as massless modes of the closed string at one-loop-string level [7]. Massive modes have received little attention in this approach. To our knowledge, only the tachyon has been considered [8,9]. The aim of this paper is to present a procedure to obtain the field equations corresponding to arbitrary modes of the bosonic string using the sigma-model approach. The treatment can be extended easily to the case of supersymmetric strings.

Most of the work done in the sigma-model approach have consisted of the analysis of the sigma-model beta functions corresponding to the renormalizable operators associated to the massless string modes. Another procedure consists of computing the renormalized sigma-model effective action and then impose conformal invariance through the requirement of the decoupling of the conformal mode i.e., the requirement that the Weyl-anomaly coefficients vanish, or, equivalently, that the trace anomaly vanishes (there are not gravitational anomalies). Both methods have been compared in [10,11] and shown not to give the same conditions beyond two loops. Explicit calculations at three loops and higher [12] of the beta function have been compared to field equations obtained from string effective action arguments [13] and shown the discrepancy between the two approaches. What should be equivalent to the string equations of motion is the condition of the absence of the Weyl anomaly. Certainly, the vanishing of the Weyl-anomaly coefficients imply the vanishing of the beta functions. The converse is not true. However, it seems likely [11] that the vanishing of the beta functions implies certain relations that when are satisfied make the Weyl-anomaly coefficients to vanish. Therefore both approaches may well be equivalent. As discussed in [11] the second approach does not contain ambiguities and so it seems more suitable to use (although the corresponding computations are in general more involved).
The inclusion of the tachyon mode in the sigma-model analysis has been carried out in [8,9]. The coupling involves a superrenormalizable operator and so one does not need other operators included in the sigma-model to be able to obtain the renormalized effective action. Similarly, the massless modes, which consists of renormalizable operators constitute a consistent set of operators which do not mix under renormalization with other operators. However, if one includes any of the higher massive modes one is forced to introduce all of them since they are non-renormalizable and they are going to mix under renormalization. The corresponding sigma model containing all the string modes is renormalizable in a generalized sense since all the operators appear in the action. No consistent truncation exists keeping only some of the modes. The aim of this paper is to provide a prescription to obtain information about the field equations (and in turn the spacetime effective action) that these massive modes must satisfy under the requirement of world-sheet conformal invariance.

We will use the second approach above, i.e., we will impose conformal invariance by demanding that the coefficients of the Weyl anomaly vanish. Since the presence of non-renormalizable operators implies the existence of counterterms with involved mixings among operators, it would be preferable to have a prescription which contains the substraction procedure built-in. Such a prescription exists and is called operator regularization (OR) [14]. The prescription provides a finite sigma model effective action which is supposed to be equivalent to the one obtained by other renormalization prescriptions in a specific scheme [14]. The advantage of this procedure is that symmetries are preserved all along and that there is no need to go through the effects of counterterms when computing the effective action. Since the Weyl-anomaly coefficients are unambiguous [11] we may well use this prescription. We would like to remark that since the prescription provides finite quantities without going through the substraction procedure it will be very useful in calculations involving theories that are renormalizable in a generalized sense as the one at hand. OR has been already used in this context [15] in dealing with the massless modes of the closed bosonic string. However, where the prescription is extremely useful is in dealing with non-renormalizable operators.

In this paper we will present the set up for the analysis of massive string modes. We will be concerned mainly with the the linearized equations of these modes, i.e., we will obtain the Weyl-anomaly coefficients up to terms linear in the background fields. Certainly, dynamical aspects must be treated in future work. We will be dealing mainly with the open string since its massive modes start at a lower spin than in the case of the closed string. However, the procedure is equally valid for the closed string as well as for the supersymmetric ones. As stressed in [8] it would be very interesting to understand what is the role of the background fields associated to different operators. In particular
one would like to know if some of them may be identified as auxiliary or Stuckelberg fields and if the resulting field equations contain enough auxiliary sector to be able to write down an effective action. These questions are addressed in this paper.

The paper is organized as follows. In sect. 2 we introduce the open bosonic string background fields and we discuss the corresponding sigma model. In sect. 3 the effective action for the second and third mass levels is computed up to terms linear in the background fields and linear in the background 2d metric using OR. The resulting field equations are presented in sect. 4 after imposing conformal invariance on the effective action, and the degrees of freedom are discussed. In sect. 5 additional field equations are obtained after considering terms quadratic in the 2d background metric in the effective action, and the complete set of equations is discussed. Finally, in sect. 6 we state our conclusions and discuss future work. In appendices A and B we present some of the details of our calculations.
2. Background Fields

In this section we present the general form of the couplings of the background fields in the sigma model. Let us start with general considerations. In what follows we will be concerned with the two-dimensional on-shell effective action $\Gamma[\Phi; X]$ corresponding to the generating functional [4]:

$$Z[\Phi; X] = \sum_{\chi} \int [D\gamma_{\alpha\beta}] \int [D\xi^i] \exp\left\{-S(\Phi; \gamma_{\alpha\beta}, X^i + \xi^i)\right\} \tag{2.1}$$

where the sum goes over different topologies and $\Phi$ represents arbitrary background fields. $\gamma_{\alpha\beta}$ are metrics of two-dimensional surfaces. The generating functional in (2.1) is a non-linear version of Polyakov’s path integral in a background-field-method fashion. Notice that the variables that represent the embedding of the two-dimensional surfaces into the $D$-dimensional world are splitted in a background part $X^i$ and a quantum part $\xi^i$. The action $S$ contains the non-linearities associated to the non-linear sigma model via the background fields $\Phi$. For the closed string,

$$S_c(\Phi; \gamma_{\alpha\beta}, X) = \int_\mathcal{M} d^2\sigma \sqrt{\gamma} \left\{ T(X) + \frac{1}{4\pi\alpha'} \gamma^{\alpha\beta} G_{ij}(X) \partial_\alpha X^i \partial_\beta X^j + \frac{1}{\alpha'} \epsilon^{\alpha\beta} B_{ij}(X) \partial_\alpha X^i \partial_\beta X^j + R_\phi(X) + \frac{1}{\alpha'^2} \gamma^{\alpha\beta} \gamma^{\gamma\delta} A_{ijkl}(X) \partial_\alpha X^i \partial_\beta X^j \partial_\gamma X^k \partial_\delta X^l + \ldots \right\} \tag{2.2}$$

where $T(X)$ corresponds to the tachyon mode, $G_{ij}(X), B_{ij}(X)$ and $\phi(X)$ correspond to the massless sector built out of the graviton, the antisymmetric tensor and the dilaton respectively, and $A_{ijkl}(X)$ is the background field associated to the first massive mode of the closed string. In (2.2) $\mathcal{M}$ represents two-dimensional surfaces without boundary. Notice that all the dimension 2 operators enter in (2.2) to have the right description of the massless level. For the first massive mode one has many more operators than the one written [8]. These operators contain up to curvature square terms. Certainly, all of them must be introduced if one wants to have a theory which is renormalizable. Some of the background fields associated to this operators constitute the auxiliary and Stuckelberg sectors of that mass level [8]. We do not write these operators explicitly here since in this work we are going to be concerned with the open string where the first massive modes start at a lower spin and therefore the calculations are not so involved. We will see that a Stuckelberg sector is present in the open string.
Although the open string is not consistent by itself, for simplicity we will consider it in this paper without close strings. Certainly, the full theory is made out of the action in (2.2) and the action of the open string with all the corresponding background fields attached at the boundary (see below). However, in this work we will be considering only tree-string level and so we can ignore the effects of the closed strings for the moment. If such is the case, the action associated to the open string is,

\[ S_o(\Phi; \gamma_{\alpha\beta}, X) = S_0(\gamma_{\alpha\beta}, X) + S_I(\Phi; \gamma_{\alpha\beta}, X), \]  

(2.3)

where

\[ S_0(\gamma_{\alpha\beta}, X) = \int_M d^2\sigma \sqrt{\gamma} \{ \frac{1}{4\pi\alpha'} \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^i \}, \]

\[ S_I(\Phi; \gamma_{\alpha\beta}, X) = \int_{\partial M} e d\tau \left\{ T(X) + \frac{1}{(\alpha')^{1/2}} i A_i(X) \frac{1}{e} \frac{d}{d\tau} X^i \right. \]

\[ + \left. \frac{1}{\alpha'} A_{ij} \frac{1}{e} \frac{d}{d\tau} X^i \frac{1}{e} \frac{d}{d\tau} X^j + \frac{1}{(\alpha')^{1/2}} i B_i(X) \frac{1}{e} \frac{d}{d\tau} \left( \frac{1}{e} \frac{d}{d\tau} X^i \right) + \ldots \right\}, \]

(2.4)

and

\[ e^2(\tau) = \gamma_{\alpha\beta}(\sigma^\gamma(\tau)) \frac{1}{e} \frac{d}{d\tau} \sigma^\alpha \frac{1}{e} \frac{d}{d\tau} \sigma^\beta \]

(2.5)

where \( \tau \) parametrizes the boundary. In (2.4) \( M \) represents two dimensional surfaces with boundary \( \partial M \). The background fields in this case correspond to the tachyon mode, \( T(X) \), the massless vector mode, \( A_i(X) \), and the first massive mode, \( A_{ij}(X) \) and \( B_i(X) \). Notice that we have introduced all the dimension 2 operators at the boundary. The symmetric tensor field \( A_{ij}(X) \) is clearly associated to a spin 2 massive field. The vector field \( B_i(X) \) must play an auxiliary or Stuckelberg role. It is simple to see that, in fact, \( B_i(X) \) is going to play the role of a Stuckelberg field. The action \( S_I \) is invariant under the \( D \)-dimensional gauge transformations:

\[ \delta A_i = (\alpha')^{1/2} \partial_i \Lambda, \]

\[ \delta A_{ij} = \frac{1}{2} (\alpha')^{1/2} \partial_{(i} \Lambda_{j)}, \]

\[ \delta B_i = - i \Lambda_i, \]

(2.6)

where the parenthesis denote symmetrization of the corresponding indices. Besides the known gauge invariance associated to the vector field \( A_i \) we observe that there is a Stuckelberg-like gauge invariance for the pair \( A_{ij} \), \( B_i \). Notice that the dimension 2
operator associated to the $B_i$ field must be introduced by renormalizability. However, if the gauge symmetry for these fields in (2.6) is preserved through the computation of the effective action we may always choose in the final field equations a gauge where the field $B_i$ vanishes.

It is now clear how to proceed in general. Given a dimension, one must introduce in $S_I$, as required by renormalizability, all the operators of that dimension with the corresponding background fields attached. Gauge symmetries can be investigated already at this stage by searching for invariances of this action. Let us analyze the next mass level. The new terms in the action are,

$$\begin{align*}
S_I &= \ldots + \int_{\partial \mathcal{M}} e d\tau \left\{ iA_{ijk} \frac{1}{e} \frac{d}{d\tau} X^i \frac{1}{e} \frac{d}{d\tau} X^j \frac{1}{e} \frac{d}{d\tau} X^k + C_{ij} \frac{1}{e} \frac{d}{d\tau} \left( \frac{1}{e} \frac{d}{d\tau} X^i \right) \frac{1}{e} \frac{d}{d\tau} X^j \right. \\
&\quad \left. + iD_i \frac{1}{e} \frac{d}{d\tau} \left( \frac{1}{e} \frac{d}{d\tau} \left( \frac{1}{e} \frac{d}{d\tau} X^i \right) \right) \right\} + \ldots
\end{align*}$$

(2.7)

which account for all the dimension 3 operators. In all the equations related to this mass level we will set $\alpha^\prime = 1$. By inspection one can verify that (2.7) is invariant under the following gauge transformations,

$$\begin{align*}
\delta A_{ijk} &= \frac{1}{6} \partial_i \Lambda_{jk}, \\
\delta C_{ij} &= 2i \Lambda_{ij} + i \partial_j \Gamma_i, \\
\delta D_i &= \Gamma_i,
\end{align*}$$

(2.8)

where $\Lambda_{ij}$ is a symmetric tensor. These symmetries imply that the symmetric part of $C_{ij}$ and $D_i$ will play the role of Stuckelberg fields. However, we can see already that there is no auxiliary sector at this level. On the other hand, it is known that for the lagrangian description of a massive spin 3 field one needs a scalar auxiliary field [16]. The fact that there is not room for this field in the action (2.7) does not necessarily mean that the approach is wrong. The field equation for this auxiliary field sets it to zero and so it may happen that one gets the right field equations in the sigma-model approach but that to build a $D$-dimensional effective action one needs to introduce an auxiliary sector. On the other hand one may hope for an approach were there are more operators in $S_I$. One possibility is to introduce operators containing ghost fields. Again, renormalizability tells us that one must introduce all possible operators. This certainly constitutes a huge amount of fields with enough room for the auxiliary sector (based on our experience in string field theory). In fact, a preliminary analysis of this approach shows that one is bound to obtain field equations associated to a BRST fixed string field theory. In this approach one does not longer look at conformal invariance but instead to BRST invariance and the nilpotency of the BRST charge.
For the case of the closed string one possesses more room in the sigma-model approach since terms with arbitrary powers of two-dimensional curvatures are available (this kind of terms are very unnatural in the case of the open string). It is very important to know which kinds of auxiliary and Stuckelberg sectors are contained in this case.
3. Effective Action at Linear Order

In this section we compute the 2d effective action associated with the generating functional (2.1) up to terms linear in the background fields and linear in the 2d metric. We will be considering all along the first two massive levels performing the calculations in full detail for the first level (together with the vector and tachyon fields) and will present the corresponding results for the second massive level at the end of the section commenting the novelties that appear there.

Consider the generating functional (2.1) with the action given in (2.3) and (2.4). Since we will be working at string tree level, the two dimensional surface $M$ in (2.4) is just the upper half-plane. Let us first say a few words on the reparametrization invariance. To deal with it, we choose a conformal gauge where the metric takes the form

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta} e^{\varphi}$$

and we then introduce the corresponding Faddeev-Popov ghosts. The generating functional (2.1) becomes

$$Z[\Phi; X] = \int [Db_{\alpha\beta}][Dc_{\alpha}][D\varphi]$$

$$\int [D\xi^i] \exp\{-S_o(\Phi; \delta_{\alpha\beta} e^{\varphi}, X^i + \xi^i) - S_{gh}(\delta_{\alpha\beta} e^{\varphi}, b_{\alpha\beta}, c_{\alpha})\}$$

where

$$S_{gh}(\gamma_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha}) = \frac{1}{2\pi} \int_M d^2 \sigma \sqrt{\gamma} \gamma^{\alpha\beta} \psi^i \nabla_{\alpha} b_{\beta\gamma}$$

Notice that in equation (3.2) it appears explicitly an integration over the conformal factor $\varphi$. This integration will become trivial at the end since the requirement of conformal invariance on the effective action will imply the decoupling of $\varphi$ from the rest of the fields. On the other hand the integration over the ghost fields is straightforward since they enter the action as free fields. The result of this integration is quadratic in $\varphi$ and will be analyzed in sect. 5, together with the other contributions to that order.

We now proceed to evaluate the remaining piece of the generating functional given by

$$\tilde{Z}[\Phi; X, \varphi] = \int [D\xi^i] \exp\{-S_o(\Phi; \delta_{\alpha\beta} e^{\varphi}, X^i + \xi^i)\}$$

up to terms linear in $\varphi$. In this computation we will use OR [14]. As argued in [15] the infrared divergences present in the computation of this effective action can
be analytically continued to zero. We will disregard the infrared divergences in our calculation, \textit{i.e}, we will consider tadpole-like integrals to be zero. We have checked that the additional terms that one would obtain in the effective action after introducing a small regulating mass do not give new conditions on the background fields. All he contributions to $\tilde{Z}[\Phi; X, \varphi]$ linear in $\varphi$ and linear in the background fields come from considering tree and one loop levels in (3.4). The prescription of OR is then to expand $S_o$ in (2.4) up to terms quadratic in the quantum fields $\xi_i$. Terms linear in $\xi_i$ will be disregarded since we are concerned with the on-shell effective action. One finds from (2.3) and (2.4):

$$S_o(\Phi; \delta_{\alpha \beta} e^{\varphi} X + \xi) = S^{(0)}(\Phi; \varphi, X) + S^{(2)}(\Phi; \varphi, X, \xi) + \cdots$$  \hspace{1cm} (3.5)

where

$$S^{(0)}(\Phi; \varphi, X) = \frac{1}{4\pi\alpha'} \int_M d^2 \sigma \partial^\alpha X \partial_\alpha X + \int_{\partial M} e d\tau \left[ T(X) + \frac{1}{(\alpha')^{1/2}} i A_i(X) \frac{d}{e d\tau} X^i \right.$$  

$$+ \frac{1}{\alpha'} A_{ij} \frac{d}{e d\tau} X^i \frac{d}{e d\tau} X^j + \frac{1}{(\alpha')^{1/2}} i B_i(X) \frac{d}{e d\tau} \left( \frac{d}{e d\tau} X^i \right) \left. \right],$$  \hspace{1cm} (3.6)

$$S^{(2)}(\Phi; \varphi, X, \xi) = \int_M d^2 \sigma e^{\varphi} \xi^i \frac{1}{2} \left( -\frac{1}{2\pi\alpha'} \delta_{ij} \Box \right) \xi^j$$  

$$+ \int_{\partial M} e d\tau \xi^i \frac{1}{2} \left[ K^{(0)}_{ij} + K^{(1)}_{ij} \frac{d}{e d\tau} + K^{(2)}_{ij} \frac{d}{e d\tau} \left( \frac{d}{e d\tau} \right) \right] \xi^j,$$  \hspace{1cm} (3.7)

and

$$e^2 = e^{\varphi(\sigma(\tau))},$$  \hspace{1cm} (3.8)

$$\Box = \frac{1}{\sqrt{\gamma}} \partial_\alpha \sqrt{\gamma} \alpha^\beta \partial_\beta,$$  \hspace{1cm} (3.9)

with

$$K^{(0)}_{ij} = \partial_i \partial_j T + \frac{1}{\alpha'^{1/2}} i \partial_i \partial_j A_k \frac{d}{e d\tau} X^k + \frac{1}{\alpha'} \partial_i \partial_j A_{kl} \frac{d}{e d\tau} X^k \frac{d}{e d\tau} X^l$$  

$$+ \frac{1}{\alpha'^{1/2}} i \partial_i \partial_j B_k \frac{d}{e d\tau} \left( \frac{d}{e d\tau} X^k \right),$$  \hspace{1cm} (3.10)

$$K^{(1)}_{ij} = 2 \left[ \frac{i}{\alpha'^{1/2}} \partial_i A_j + \frac{2}{\alpha'} \partial_i A_{kj} \frac{d}{e d\tau} X^k - \frac{1}{\alpha'} \partial_k A_{ij} \frac{d}{e d\tau} X^k \right],$$  \hspace{1cm} (3.11)

$$K^{(2)}_{ij} = 2 \left[ \frac{i}{\alpha'^{1/2}} \partial_i B_j - \frac{1}{\alpha'} A_{ij} \right].$$  \hspace{1cm} (3.12)

The expansion of the action could have been done in such a way that each of the $K_{ij}$’s defined previously is gauge invariant. However, we have chosen them not to
be gauge invariant in order to have a check of our results. Since the procedure we
are using respects all the symmetries of the problem, it is reassuring to find that the
final set of equations is indeed gauge invariant. To carry out the calculation in this
way provides a way to verify the coefficients associated with $K^{(0)}$ and $K^{(1)}$ in the
calculation of the effective action (see below). Since $K^{(2)}$ is always a gauge invariant
combination its coefficient remains unchecked from the analysis at this level. However,
upon consideration of higher massive levels, this coefficient will also be checked using
gauge invariance.

To deal with the complications of having an action with some parts integrated over
the full manifold (upper half-plane) and others integrated only over the boundary, we
will define extensions of the integrals over the boundary to the whole surface in such a
way that $S^{(2)}$ in (3.7) becomes

$$\int_{\mathcal{M}} d^2\sigma e^{\phi/2} \frac{1}{2} \xi^i \left[ -\frac{1}{2\pi\alpha'} \delta_{ij} \square + M_{ij} + M_{ij}^\alpha \nabla_\alpha + M_{ij}^{\alpha\beta} \nabla_\alpha \nabla_\beta \right] \xi^j. \tag{3.13}$$

The objects $M_{ij}$, $M_{ij}^\alpha$ and $M_{ij}^{\alpha\beta}$ are given by

$$M_{ij}(\tau, \sigma) = e^{-\varphi/2} \delta(\sigma) K_{ij}^{(0)}(\tau), \tag{3.14}$$
$$M_{ij0}(\tau, \sigma) = e^{-\varphi/2} \delta(\sigma) e^{\varphi/2} K_{ij}^{(1)}(\tau), \tag{3.15}$$
$$M_{ij1}(\tau, \sigma) = 0, \tag{3.16}$$
$$M_{ij00}(\tau, \sigma) = e^{-\varphi/2} \delta(\sigma) e^{\varphi/2} e^{\varphi/2} K_{ij}^{(2)}(\tau), \tag{3.17}$$
$$M_{ij01}(\tau, \sigma) = M_{ij10}(\tau, \sigma) = M_{ij11}(\tau, \sigma) = 0, \tag{3.18}$$

where we have chosen $\sigma_0$ to be the coordinate along the boundary $\tau = \sigma_0$ and $\sigma = \sigma_1$.

To obtain (3.13) one must use the fact that the field $\varphi$ is such that $\partial_{\sigma} \varphi = 0$ at the
boundary, i.e., at $\sigma = 0$.

Once we have organized the action as in (3.13) we can proceed to compute (3.4). We
will use the technique of OR and a weak field expansion in the field $\varphi$. This technique
is described in appendix A as well as the details of the calculation that now we outline.
At order linear in $\varphi$ and in the background fields, we need to compute essentially the
determinant of the operator present in (3.13):

$$\det^{-\frac{1}{2}} \left[ -\frac{1}{2\pi\alpha'} \delta_{ij} \square + M_{ij} + M_{ij}^\alpha \nabla_\alpha + M_{ij}^{\alpha\beta} \nabla_\alpha \nabla_\beta \right]. \tag{3.19}$$

The reason for this is that at order linear in $\varphi$ and in the background fields there are
only 1-loop contributions (see appendix A). This calculation is carried out in appendix
A starting after (A13). The previous part of appendix A deals with a short review of OR. The resulting contribution to the effective action is given in (A34):

\[ \Gamma^{(1)} = \frac{\alpha'}{2} \int_{\mathcal{M}} d^2 \sigma \varphi(\sigma) [M_{ii}^{(0)}(\sigma) - \frac{1}{2} \partial_\alpha M_{ii}^{(0)\alpha}(\sigma) - \frac{1}{3} (\frac{1}{4} \partial^2 \delta_{\alpha\beta} - \partial_\alpha \partial_\beta) M_{ii}^{(0)\alpha\beta}(\sigma)], \]

(3.20)

where \( M_{ij}(\sigma), M_{ij}^{(0)\alpha}(\sigma) \) and \( M_{ij}^{(0)\alpha\beta} \), are the terms of order zero in powers of \( \varphi \) in (3.14)-(3.18) and a sum over \( i \) is understood. In fact using those expressions we can write \( \Gamma^{(1)} \) as an integral over the boundary,

\[ \Gamma^{(1)} = \frac{\alpha'}{2} \int_{\partial \mathcal{M}} d\tau \varphi(\sigma(\tau)) \left[ K_{ii}^{(0)}(\tau) - \frac{1}{2} \dot{K}_{ii}^{(1)}(\tau) + \frac{1}{4} \ddot{K}_{ii}^{(2)}(\tau) \right], \]

(3.21)

where the dot means derivation with respect to \( \tau \). Extracting from (3.6) the tree level contribution to the effective action which is linear in \( \varphi \), we get

\[ \Gamma = \Gamma^{(0)} + \Gamma^{(1)} = \frac{1}{2} \int_{\partial \mathcal{M}} d\tau \varphi(\sigma(\tau)) \left[ K(\tau) + \alpha' \left( K_{ii}^{(0)}(\tau) - \frac{1}{2} \dot{K}_{ii}^{(1)}(\tau) + \frac{1}{4} \ddot{K}_{ii}^{(2)}(\tau) \right) \right], \]

(3.22)

where

\[ K = T - \frac{1}{\alpha'} A_{ij} \dot{X}^i \dot{X}^j + \frac{i}{\alpha'^{1/2}} \dot{\omega}_i B_j \dot{X}^i \dot{X}^j. \]

(3.23)

Notice that in \( \Gamma^{(0)} \) we have performed an integration by parts to factor out \( \varphi \). In the next section we will extract the field equations encoded in this effective action.

Although the calculation has been presented in a conformal gauge, it is worth noting that it can equally well be done in a fully covariant manner with the complete metric using a weak field expansion. In that case we get:

\[ \Gamma^{(1)}_{\text{cov}} = \frac{\alpha'}{2} \int_{\mathcal{M}} d^2 \sqrt{\gamma} \left[ M_{ii}(\sigma) - \frac{1}{2} \nabla_\alpha M_{ii}^{\alpha}(\sigma) - \frac{1}{3} (\frac{1}{4} \delta_{\alpha\beta} - \nabla_\alpha \nabla_\beta) M_{ii}^{\alpha\beta}(\sigma) \right] \]

\[ \int_{\mathcal{M}} d^2 \sqrt{\gamma'} G(\sigma, \sigma') R(\sigma'), \]

(3.24)

where \( R \) is the world-sheet scalar curvature and \( G \) is the propagator in the upper
half-plane,
\[ \Box G(\sigma, \sigma') = \frac{1}{\sqrt{\gamma}} \delta^{(2)}(\sigma - \sigma'), \]
\[ n^\alpha \nabla_\alpha G(\sigma, \sigma') = 0 \bigg|_{\partial \mathcal{M}}, \]
with \( n^\alpha \) being a vector normal to the boundary. Notice that (3.24) coincides with (A34) in the conformal gauge.

All the procedure can be carried out similarly for the third excited level whose action and gauge transformations are given in (2.7) and (2.8). The result is
\[ \Gamma = \frac{1}{2} \int_{\partial \mathcal{M}} d\tau \varphi \left[ \tilde{K} + \alpha' \left( \tilde{K}^{(0)}_{ii} - \frac{1}{2} \tilde{K}^{(1)}_{ii} + \frac{1}{4} \tilde{K}^{(2)}_{ii} - \frac{1}{8} \tilde{K}^{(3)}_{ii} \right) \right], \tag{3.26} \]
where the \( \tilde{K} \)'s are now
\[ \tilde{K} = (-iA_{ijk} + \frac{1}{2} \partial_i C_{jk} - i \frac{1}{2} \partial_i \partial_j D_k) \dot{X}^i \dot{X}^j \dot{X}^k + \frac{1}{2} (-C_{ij} + i \partial_j D_i) \dot{X}^i \dot{X}^j + \frac{1}{2} (C_{ij} - i \partial_j D_i) \ddot{X}^i \ddot{X}^j, \tag{3.27} \]
\[ \tilde{K}^{(0)}_{ij} = i \partial_i \partial_j A_{klm} \dot{X}^k \dot{X}^l \dot{X}^m + \partial_i \partial_j C_{kl} \dot{X}^k \dot{X}^l + i \partial_i \partial_j D_k \ddot{X}^k, \tag{3.28} \]
\[ \tilde{K}^{(1)}_{ij} = 6i (\partial_i A_{klj} - \partial_l A_{kij}) \dot{X}^k \dot{X}^l - 6i A_{kij} \ddot{X}^k + 2 \partial_l C_{kj} \ddot{X}^k, \tag{3.29} \]
\[ \tilde{K}^{(2)}_{ij} = -6i A_{kij} \dot{X}^k + 2 (\partial_i C_{jk} - \partial_k C_{ji}) \ddot{X}^k, \tag{3.30} \]
\[ \tilde{K}^{(3)}_{ij} = 2 (-C_{ji} + i \partial_i D_j). \tag{3.31} \]
The only new feature in the calculation comes from the piece with three derivatives in (3.26), whose coefficient is given by OR. The previous result can be used for the rest of the action.
4. Preliminary Field Equations

In this section we discuss the equations resulting from the requirement of conformal invariance of the action given in (3.22). This requirement is equivalent to ask independence of $\varphi$, i.e.,

$$K + \alpha'(K_{ii}^{(0)} - \frac{1}{2} \dot{K}_{ii}^{(1)} + \frac{1}{4} \ddot{K}_{ii}^{(2)}) = 0. \quad (4.1)$$

Using the definitions of the $K$’s in (3.23) and (3.10)-(3.12) and decomposing in independent pieces we find the following equations

$$(\Box + \frac{1}{\alpha'})T = 0, \quad (4.2)$$

$$\Box A_i - \partial_i \partial^k A_k = 0, \quad (4.3)$$

$$\Box - \frac{1}{\alpha'} A_{ij} - \partial_i \partial^k A_{kj} + \frac{1}{2} \partial_i \partial_j A + \frac{i}{2} \alpha^{1/2} \partial_i B_j \partial^k A_{ik} - \frac{1}{2} \partial_i A = 0, \quad (4.4)$$

$$\Box B_i + \frac{1}{2} \partial_i \partial^k B_k + \frac{i}{\alpha^{1/2}} (2 \partial^k A_{ik} - \frac{1}{2} \partial_i A) = 0, \quad (4.5)$$

where $A = A'_{i}$. We would like to make a few comments regarding these equations. First, we observe that the approach is going to give the right values for the masses corresponding to the different levels since the kinetic part $\Box$ comes from $K_{ii}^{(0)}$, as in (3.10) from terms like the first three. The indices of the two derivatives are going to be always contracted and so for all the fields we will have a $\Box$ with the same factor in front. The value of the mass comes from the appearance of the field in $K$ (see (3.23)). Here we see that there are two possibilities. Either the field gets a numerical factor which depends on the number of times that $\frac{1}{e}$ is multiplying the field or, due to the integrations by parts, that are needed to factor out $\varphi$, one gets derivatives of the fields. In the second case one is dealing with fields that play the role of Stuckelberg fields. In the first case, that numerical factor is going to be multiplying the basic value of the mass squared, $\frac{1}{\alpha'}$. In (4.2) this value is +1 because there is only one $e$ in the numerator in (3.6). The vector field equation (4.3) does not have mass term because there is not $e$ factor for that field in (3.6). For the first massive level the sign of the mass squared changes in (4.4) since now there is a global factor $e$ in the denominator concerning the field $A_{ij}$ in (3.6). The equation for the Stuckelberg field $B_i$ shows the behavior announced: no mass term. One can trace back the reason for this in (3.6) where one can see that after integrating by parts to factor out $\varphi$ the terms which could lead to a mass term for $B_i$ cancel out. This behavior is general for any mass level.
Another important property of equations (4.2)-(4.5) is the appearance of terms with three derivatives already at this level. This is a consequence of the fact that we are dealing with non-renormalizable terms in the sigma model action.

The same equations are obtained from the covariant action given in (3.24). There one must perform a Weyl transformation

$$\delta \gamma_{\alpha \beta} = 2 \Lambda(\sigma) \gamma_{\alpha \beta}, \quad (4.6)$$

restricted by the boundary condition

$$n^\alpha \nabla_\alpha \Lambda(\sigma) = 0 \bigg|_{\partial M}, \quad (4.7)$$

and use

$$\delta \sqrt{\gamma} R = 2 \sqrt{\gamma} \Box \Lambda, \quad \delta e = e \Lambda. \quad (4.8)$$

The equations obtained in this way are just the ones compiled in (4.1).

For the next massive level one gets the same equation as in (4.1) supplemented by the addition of ($\alpha' = 1$)

$$-\frac{1}{8} \tilde{K}_{ji}^{(3)}, \quad (4.9)$$

where $\tilde{K}_{ji}^{(3)}$ is given in (3.31).

Let us analyze the equations obtained in (4.2)-(4.5). These equations are invariant under the gauge transformations (2.6). (4.2) is the linearized equation of the tachyon and (4.3) corresponds to Maxwell’s equations. The other two equations, (4.4) and (4.5), correspond to the first massive level of the open string at the linearized level. This linearized equations are enough to analyze the degrees of freedom that are represented by these fields. In order to do this we choose the gauge where the Stuckelberg field $B_i$ is set to zero. In this gauge (4.4) and (4.5) become

$$(\Box - \frac{1}{\alpha'}) A_{ij} - \partial_i \partial^k A_{k ij} + \frac{1}{2} \partial_i \partial_j A = 0, \quad (4.10)$$

$$\partial^k A_{ki} - \frac{1}{4} \partial_i A = 0. \quad (4.11)$$

The degrees of freedom corresponding to a massive particle of spin two are described by a symmetric tensor satisfying besides the on-shell condition $$(\Box - \frac{1}{\alpha'}) A_{ij} = 0$$ the condition of transversality ($\partial^k A_{ki} = 0$) and tracelessness ($A = 0$). Not all these
conditions can be derived from (4.10) and (4.11). In fact, plugging (4.11) into (4.10), these equations become,

\[(\Box - \frac{1}{\alpha'})A_{ij} = 0, \quad \partial^k A_{ki} - \frac{1}{4} \partial_i A = 0,\]  

(4.12)

\[\text{i.e. they represent the degrees of freedom of a massive spin two field plus a scalar, say, the trace of } A_{ij}.\]  

Actually, this is what one should expect just considering only terms linear in \(\varphi\) in the effective action. In the analysis of physical states using the Virasoro conditions one finds that at this mass level one has the degrees of freedom corresponding to a particle of spin two only if the space-time dimension is 26. Otherwise one has an extra degree of freedom. One would expect that in our approach the dimension of space-time plays an important role in the analysis of the degrees of freedom associated to this first mass level. This is in fact the case. As we will describe in the next section, conformal invariance at order \(\varphi^2\) implies that the space-time dimension must be 26 and that the trace of \(A_{ij}\) must vanish. Therefore, at this level there are only the physical degrees of freedom corresponding to a particle of spin two. This fact is general for any higher mass level, \(i.e.,\) the order \(\varphi^2\) in the effective action gives an additional contribution to the field equations obtained from the linear order in \(\varphi\) which makes the counting of degrees of freedom as the one obtained from the Virasoro conditions in 26 dimensions. The equations obtained from the linear order in \(\varphi\) contain a number of physical degrees of freedom equal to the one obtained from the Virasoro conditions when the dimension of space-time is not 26. For the massless and tachyonic sectors, one finds at order \(\varphi^2\) expressions which vanish if one uses the field equations obtained at the linear level in \(\varphi\) and therefore the number of degrees of freedom obtained from the linear analysis is the right one. As we have seen, this is not the case for the non-renormalizable mass levels, in agreement with the fact that the space-time dimension must play a special role in the counting of degrees of freedom for these levels.

Let us perform this analysis in the next mass level. The field equations at linear level in \(\varphi\) come from (3.26) and (3.27)-(3.31) \((\alpha' = 1),\)

\[
(\Box - 2)A_{ijk} - \frac{1}{2} \partial_{(i} \partial^j A_{jk)}l + \frac{1}{4} \partial_{(i} \partial_j \hat{A}_{k)} - \frac{i}{6} \partial_{(i} B_{jk)}
- \frac{1}{6} \partial_{(i} \partial_j D_{k)} - \frac{i}{12} \partial_{(i} \partial_j \partial^l B_{k)}l + \frac{i}{12} \partial_{(i} \partial_j \partial^l X_{k)}l
+ \frac{i}{4} \partial_{(i} \partial_j \partial_k B - \frac{1}{4} \partial_{(i} \partial_j \partial_k \partial^l D_l = 0, \]

(4.13)

\[B_{ij} + \frac{1}{4} \partial^l \partial_{(i} B_{j)}l - \frac{3}{4} \partial_{(i} \partial_j B - 6i \partial^l A_{li}j + \frac{9i}{4} \partial_{(i} \hat{A}_{j)}
- \frac{5}{4} \partial^l \partial_{(i} X_{j)}l - \frac{3i}{4} \partial_{(i} \partial_j \partial^l D_l = 0, \]

(4.14)
\[(\Box - 2)X_{ij} - \frac{3i}{4} \partial_{[i} \hat{A}_{j]} - \frac{3}{4} \partial^l \partial_{[i} X_{ij]l} + \frac{1}{4} \partial_{[i} \partial^l B_{j]} + i \partial_{[i} D_{j]} = 0, \quad (4.15)\]

\[\Box D_i - \frac{1}{4} \partial_t \partial^l D_{i l} + \frac{3}{2} \hat{A}_i + \frac{i}{2} \partial_t B_{i} - \frac{3i}{2} \partial^l X_{i l} + \frac{i}{4} \partial_l B = 0, \quad (4.16)\]

where we have defined

\[B_{ij} = \frac{1}{2} C_{(ij)}, \quad X_{ij} = \frac{1}{2} C_{[ij]}, \quad \hat{A}_i = A_{ji}, \quad B = B_i^i. \quad (4.17)\]

These equations are invariant under the set of gauge transformations (2.8). This allows us to set the Stuckelberg fields to zero

\[B_{ij} = 0, \quad D_i = 0, \quad (4.18)\]

and obtain the following equations

\[(\Box - 2)A_{ijk} - \frac{1}{2} \partial_{(i} \partial^l A_{kj])l + \frac{1}{4} \partial_{(i} \partial_j \hat{A}_k) + \frac{i}{12} \partial_{(i} \partial_j \partial^l X_{k]l} = 0, \quad (4.19)\]

\[(\Box - 2)X_{ij} - \frac{3}{4} \partial^l \partial_{[i} \hat{A}_{j]}\l + \frac{3i}{4} \partial_{[i} \hat{A}_{j] = 0, \quad (4.20)\]

\[\partial^l A_{lij} - \frac{3}{8} \partial_{(j} \hat{A}_{i)} - \frac{5i}{24} \partial^l \partial_{(i} X_{j]l} = 0, \quad (4.21)\]

\[i \hat{A}_i + \partial^l X_{i l} = 0. \quad (4.22)\]

These equations contain the type of degeneracy discussed above. In fact, one can verify that introducing (4.21) in (4.19) and (4.22) in (4.20), they are equivalent to

\[(\Box - 2)A_{ijk} = 0, \quad (4.23)\]

\[(\Box - 2)X_{ij} = 0, \quad (4.24)\]

\[\partial^l A_{lij} - \frac{3}{8} \partial_{(j} \hat{A}_{i)} - \frac{5i}{24} \partial^l \partial_{(i} X_{j]l} = 0, \quad (4.25)\]

\[i \hat{A}_i + \partial^l X_{ji} = 0. \quad (4.26)\]

Let us count the physical degrees of freedom involved in these equations. The components of the tensors $A_{ijk}$ and $X_{ij}$ are $\frac{1}{2} D(D + 1)(D + 2) + \frac{1}{4} D(D - 1)$. Equations (4.23) and (4.24) are just the on-shell conditions. Equations (4.25) and (4.26) give us $\frac{1}{2} D(D + 1) + D - 1$ independent conditions (notice that the trace of (4.25) can be
derived from (4.26)). Therefore, since the content of this level in $D = 26$ must be the corresponding to a massive particle with three-index tensor polarization symmetric in all its indices and traceless ($\frac{1}{6}(D - 1)D(D + 1) - (D - 1)$ degrees of freedom) and a another massive particle with two-index tensor polarization antisymmetric in its indices ($\frac{1}{2}(D - 1)(D - 2)$ degrees of freedom), we have $D - 1$ extra degrees of freedom in the equations (4.23)-(4.26). One can verify that in fact these are the extra degrees of freedom that one would have moving away from $D = 26$ in the analysis of the Virasoro conditions at this mass level. In the next section we will observe that the terms of order $\varphi^2$ in the effective action are going to introduce new conditions on the background fields in such a way that in $D = 26$ the extra degrees of freedom disappear.
5. Effective Action at Quadratic Order and Complete Field Equations

In this section we will analyze the effective action corresponding to the generating functional (3.2) at quadratic order in $\varphi$. From the new term obtained, after imposing conformal invariance, we will find a set of additional equations which are just the right ones to remove the extra degrees of freedom found when taking into account only the effective action at linear order in $\varphi$. This kind of phenomena has not been observed before and makes the behavior of the massive levels different from that of the tachyon and massless levels in the $\sigma$-model approach. On the other hand, one should expect a behavior of this kind just from the patterns observed in the analysis of states looking at the Virasoro conditions. Namely, one observes there that for the massive levels one has states whose norm vanishes only in 26 dimensions. Since the anomaly contribution is order $\varphi^2$ one should expect that at this order more field equations are obtained in such a way that the extra degrees of freedom present at linear order are removed.

The first contribution at order $\varphi^2$ we have to deal with is the anomaly itself. It comes when considering order $\varphi^2$ in the effective action and zero order in the background fields. The result is very well known,

$$\Gamma_{an}^{(2)} = \frac{D - 26}{48\pi} \int_M d^2 \sigma \varphi(\sigma) \delta^2 \varphi(\sigma), \quad (5.1)$$

or, in covariant form,

$$\Gamma_{an}^{(2)} = \frac{D - 26}{48\pi} \int_M d^2 \sigma \sqrt{\gamma} \int_M d^2 \sigma' \sqrt{\gamma'} R(\sigma) G(\sigma, \sigma') R(\sigma'), \quad (5.2)$$

where $G(\sigma, \sigma')$ is the propagator (3.25). In (5.1), the contribution proportional to $D$ comes from the bosonic part in (3.2), and the other one from the ghost action (3.3). The bosonic contribution has been computed in the context of OR in [15].

The contribution (5.1) for the effective action stands by itself since the contributions coming from the term linear in background fields are all written as an integration over the boundary. Therefore, one must have $D = 26$ from conformal invariance.

Let us now consider the contributions at order $\varphi^2$ and linear in background fields. At this order there are one and two-loop contributions. On the other hand there are several independent structures at this order among the possible contributions to the effective action. We will not present here the calculations regarding all of them. We will concentrate on one independent structure and we will extract the field equations
that it originates. The rest of the structures vanish consistently when one uses the full set of field equations. To better argue why we choose a particular structure let us look first at the effective action from a covariant point of view. Since the arbitrary mass parameter $\mu$ (see (A23) and below) drops out in the effective action we expect (see (3.24)) schematically, terms like $R^\frac{1}{2}R M^\alpha\alpha$, $\partial_\alpha\partial_\beta R^\frac{1}{2}RM^\alpha\beta$, $\partial_\alpha R^\frac{1}{2}RM^\alpha$, or $R^\frac{1}{2}RM$. Certainly, the first one is independent from the rest and carry $M^\alpha\alpha$ without any derivative. Looking at (3.17) and (3.12), we see that to make $M^\alpha\alpha$ vanish would fulfill our purposes since then the trace of $A_{ij}$ will be zero that is all we need to have the right number of degrees of freedom. We see then that there is a term that is likely to appear in the effective action and which would give us the right physics. Notice also that such a term is very close to the form (5.2). Let us try to do a similar analysis with the structures present in the conformal gauge. In this case it is not hard to convince one-self that, again, under the assumption that the arbitrary mass $\mu$ drops out, the only possible structures in the integration over the boundary are $\varphi\varphi$, $\varphi\partial_\tau\varphi$, $\varphi\partial_\sigma^2\varphi$, $\varphi\partial_\sigma\partial_\varphi$ and $\varphi\partial_\sigma\varphi$ multiplying some of the $K$’s or $\tau$-derivatives of the $K$’s. We will concentrate in the last structure. It is certainly one of the closest to the one described above in the covariant analysis and the one that is likely to have the smallest number of contributions.

There are three types of contributions at order $\varphi^2$. Two types are coming from one-loop, one involving three interaction operators ($H_I$, see (A13)), and another one involving two of these operators. The third type corresponds to contributions coming from two loops.

We will start with the first type. These are the hardest to compute. One needs to obtain the contributions of this type to the determinant (3.19). Among all the possible terms of this type one is interested only in those which lead to the structure $\partial_\sigma\varphi\partial_\sigma\varphi$. In momentum space these come from pieces which involve $p \cdot q$ (notice that now we have three functions in the effective action and hence one has two momenta). Certainly, having two momenta floating around one could have in the effective action more complicated structures, like $\log\left(\frac{p^2}{p \cdot q}\right)$, for example (remember that the mass $\mu$ is supposed to drop out and it will). However, thinking about the action in configuration space, since we do not have a mass parameter at hand and we expect general covariant expressions there is not room for terms like logarithms, etc.

If the arguments above are correct one would expect a miraculous behavior of the one-loop integrations in such a way that the final expression is simply polynomial. In the appendix B this computation is carried out and, in fact, this extraordinary behavior is observed. A very complicated integral turns out to give a very simple result. We have been able to verify this numerically. We do not possess any analytic method to
perform the integral. The contribution from this first type turns out to be very simple. From (B16), (3.17) and (3.18),

\[ \Gamma_1^{(2)} = \frac{1}{16} \int_{\partial M} \partial \sigma \varphi \partial_{\alpha} \varphi K_{ii}^{(2)}, \]  

where we have dropped the first term in (B16) since it does not contribute to the selected structure. The second type of contributions comes from the higher order terms in the expansion (A17) when only two \( H_I \) are considered in (A13). For this computation we can use the result (A34) which tells us that for the selected structure we have to concentrate our attention in \( M_{ij}^{(\alpha \beta)} \). There are two kinds of contributions of this type. One correspond to the case when one \( H_I \) is \( -\varphi \partial^2 \) (see (A15)) and the other is taken from the expansion (A17),

\[ H_{ij} = \ldots -2M_{ij}^{(0)\alpha \beta} \varphi \partial_\alpha \partial_\beta + M_{ij}^{(0)\alpha \beta} (\partial_{\alpha} \varphi) \partial_\beta + M_{ij}^{(1)\alpha \beta} \partial_\alpha \partial_\beta + \ldots \]  

where \( M_{ij}^{(1)\alpha \beta} \) is the part of (3.17) and (3.18) at order linear in \( \varphi \). The other kind corresponds to one \( H_I \) being \( \frac{1}{2} \varphi^2 \partial^2 \) (see (A15)), and the other \( H_I \) being the last term in (A17). The contribution from both kinds can be simply written using (A34). It turns out that there is contribution to the selected structure only from the second kind,

\[ \Gamma_{II}^{(2)} = \frac{1}{24} \int_{\partial M} \partial \sigma \varphi \partial_\sigma \varphi K_{ii}^{(2)}. \]  

Finally, we have to deal with the third type of contributions, the ones appearing from two loops. In the last part of appendix B it is shown that there are not contributions to the selected structure at two loops. Therefore, the sum of (5.3) and (5.5) is the full answer:

\[ \Gamma^{(2)} = \Gamma_1^{(2)} + \Gamma_{II}^{(2)} = \frac{5}{48} \int_{\partial M} \partial \sigma \varphi \partial_\sigma \varphi K_{ii}^{(2)}.\]  

Imposing conformal invariance one obtain one more field equation,

\[ K_{ii}^{(2)} = 0. \]
level, in the gauge where the Stuckelberg field vanishes, is

\[
(\Box - \frac{1}{\alpha'}) A_{ij} = 0, \\
\partial^i A_{ij} = 0, \\
A = A^i_i = 0,
\]

and therefore, at this level the description corresponds to a massive particle with a two index symmetric and traceless polarization tensor.

For the next level one has to compute the contribution to the selected structure from the term (3.31). One finds, after a similar calculation to the previous one that conformal invariance implies,

\[
\tilde{K}^{(2)}_{ii} - \frac{3}{2} \tilde{K}^{(3)}_{ii} = 0.
\]

Notice that this equation is gauge invariant under the transformations (2.8). In the gauge where the Stuckelberg fields vanish, we find, from (3.30) and (3.31),

\[
i \hat{A}_i - \frac{2}{3} \partial^j X_{ji} = 0
\]

which provides \(D - 1\) new independent conditions with respect to equations (4.23)-(4.26). The final set of equations for this level becomes \((\alpha' = 1)\),

\[
(\Box - 2) A_{ijk} = 0, \\
\partial^i A_{ijk} = 0, \\
\hat{A}_j = A^i_{ij} = 0,
\]

and

\[
(\Box - 2) X_{ij} = 0, \\
\partial^i X_{ij} = 0,
\]

which are the ones corresponding to massive particles with polarization tensors of three (traceless) symmetric indices and two antisymmetric indices, respectively.

We noticed in sect. 3 that the \(K^{(n)}\)’s in (3.10)-(3.12), and the \(\tilde{K}^{(n)}\)’s in (3.28)-(3.31) could have been selected in such a way that they are invariant under the gauge transformations (2.6) and (2.8) respectively. We made a non-invariant choice to be
able to check our calculation. In fact, all the field equations obtained turned out to be gauge invariant, showing the correctness of the numerical coefficients obtained in the computation. One could exploit this gauge invariance to learn about the field equations performing a minimum of field theory calculations. For example, the numerical coefficients appearing in (4.1) could be computed demanding gauge invariance if the $K^{(n)}$’s are chosen to be gauge variant. Similarly for higher $K$-series as the one starting as in (5.11). Since the number of gauge invariances grows with the mass level, given a coefficient in one of the $K$-series one can always take into consideration in the $K$’s enough higher-mass fields to determine its value. This is clear from the behavior observed in the levels considered in this paper. Taking into account the first massive level, it is not possible to determine all the coefficients (up to a global one which fixes the mass) in the part within parenthesis in (4.1). The last one can not be fixed because $K_{ij}^{(2)}$ can not be chosen gauge variant at this mass level. However, considering the contributions to the $K$’s from the next massive level the coefficient can be determined. One could go even further exploiting gauge invariance. It is likely that at any order in powers of the $K$’s, say, the $K^n$-series, one could fix a given numerical coefficient just imposing gauge invariance related to fields up to a given mass. Of course, the global coefficient of each series must be determined performing a field-theoretical calculation. This global factor could be obtained just looking at one simple mode, say the tachyon or the vector field. Though the gauge invariance associated to Stuckelberg fields could seem unimportant when noticed in (2.6) and (2.8), these observations show that it may be a very powerful tool in obtaining field equations.
6. Conclusions and Prospects

In this paper we have presented a prescription to compute the Weyl-anomaly coefficients corresponding to a sigma-model action containing operators of arbitrary dimension. We have worked at order linear in background fields and, after imposing conformal invariance, i.e., the vanishing of the Weyl-anomaly coefficients, we have found the field equations corresponding to the first four levels of the open string (tachyon, vector and first two massive levels). It turns out that the way the field equations are obtained for the massive levels has a different pattern than for the mass levels corresponding to renormalizable and superrenormalizable operators. Namely, one finds that keeping only the conditions obtained from the Weyl-anomaly coefficients obtained at linear level in the metric ($\varphi$) one has extra degrees of freedom. These extra degrees of freedom are removed when the Weyl-anomaly coefficients quadratic in the metric are considered. This behavior is a remnant of the pattern found when analyzing the string states using the Virasoro conditions. The extra degrees of freedom appearing in our approach when keeping only the linear part in the metric of the Weyl-anomaly coefficients are in equal number with the degrees of freedom that in the analysis of Virasoro conditions have zero norm only in 26 dimension.

Our study has shown that Weyl-anomaly coefficients have a very simple structure and that therefore are simpler to compute than one would expect. We have been arguing all along the paper that since the mass parameter corresponding to the renormalization point of the theory does not enter in these coefficients, and since they should have general covariant expressions, the form of these coefficients must be very simple. It is far from obvious that this is going to be the case when one looks at the expressions involved when dealing with loop computations. Think for instance in the integral of Schwinger parameters found in (B10) with the polynomials of the integrand shown in (B5)-(B7). The formal arguments led us to conjecture an answer for that integral that turned out to be true after carrying out a numerical integration. This case shows that the Weyl anomaly coefficients may be simpler to compute than one may have thought since their structure is very much constrained.

As was noticed at the end of sect. 5, we have observed a very promising way of obtaining field equations associated to higher-mass modes from the knowledge of the field equation of a given lower mass level. It seems likely that the gauge invariance associated to Stuckelberg fields can be exploited to determine the coefficients of the $K$-series introduced there up to a global factor.

We have not discussed in the text the existence of lagrangians leading to the field equations obtained but it is clear that they can be written out. Let us consider,
for example, the level containing $A_{ij}$ and $B_i$. In the gauge where $B_i = 0$ the field equations are (5.8)-(5.10). These equations are certainly obtained from the Fierz-Pauli lagrangian [17]. Performing the replacement $A_{ij} \rightarrow A_{ij} - \frac{i}{2}(\alpha')^{1/2}\partial_i B_j$ one obtains a lagrangian that is certainly invariant under the gauge transformations (2.6) and whose field equations are equivalent to (4.4), (4.5) and (5.7). Similarly occurs with the next level. However, in this case, in order to get a lagrangian leading to equations (5.13)-(5.17) one must introduce an auxiliary scalar field [16]. As discussed at the end of sect. 2, there is not enough room in our formulation for such a field. Nevertheless, it may well happen that this auxiliary field is only needed to build the lagrangian and that always its field equation sets it to zero. The real test of all these facts concerning lagrangians will be the analysis of the Weyl-anomaly coefficients at order higher than linear in the background fields.

Going back to the analysis of the linear order in the background fields there is still a very important question to answer. It is not still known if one has enough background fields to have a satisfactory set of field equations for each mass level. We have already seen that some auxiliary sector is definitely not present. However, this sector presumably has a field equation that sets it to zero, even at the non-linear level. We may ask if except for such an auxiliary sector there is enough background fields to have a satisfactory description. This question may be addressed just analyzing the linear level in background fields. In the case of the closed string we have learned that one needs to take into account background fields attached to the world-sheet curvature. An important open question is to verify that this is enough for a satisfactory description of the first massive level of the closed string.

For the open string, a comparison between the structure of vertex operators and the background fields appearing at the third massive level seems to indicate that one may not have enough fields to obtain a field theoretical description of all the modes. If there is not any new two-dimensional structure entering at this level either the background fields do not describe all the physical degrees of freedom, or one of the degrees of freedom if described via a combination of traces (and perhaps double traces and derivatives) of background fields. If the first possibility holds this approach does not have enough field content and one is forced to introduce operators containing ghost fields in a formalism as the one discussed at the end of sect. 2. If, on the contrary, it is the second possibility the one that turns out to be true, the approach may well be correct. However, in this case it would be necessary to introduce more auxiliary fields than just the type observed in the analysis of the second massive level when dealing with the construction of the lagrangian. Namely, it would be needed an additional scalar auxiliary field to play the role of the part of the background fields which would
have turned out to be another physical degree of freedom. Notice that also in this case the field equation of such new type of auxiliary field would set it to zero. Certainly, the construction of the lagrangian could be done for the linearized theory. Whether or not such a construction could be carried out in the non-linear theory is an open problem. To analyze which one of the two possibilities above holds is a very important question that we hope to address in future work.

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APPENDIX A

In this appendix we sketch how the operator regularization (OR) works, closely following [14], and give the details of the computation of the effective action considered in sect. 3. The general scheme will be used in appendix B to show the two-loop order calculation. OR works in the context of path-integral quantization with background field methods. In the general case a field $X^i(\sigma)$ is splitted into a classical part $X^i(\sigma)$ and a quantum part $\xi^i(\sigma)$,

$$X^i(\sigma) \rightarrow X^i(\sigma) + \xi^i(\sigma). \quad (A1)$$

OR provides a prescription to compute the on-shell renormalized generating functional for Green functions

$$Z[X, J] = \int [d\xi^i] \exp \left\{ - \int d^2 \sigma \left( \mathcal{L}(X^i + \xi^i) + J_i \xi^i \right) \right\}, \quad (A2)$$

where $J_i$ is a source and the lagrangian $\mathcal{L}$ adopts the general expansion,

$$\mathcal{L}(X^i + \xi^i) = \mathcal{L}(X^i) + N_i(X^i)\xi^i + \frac{1}{2} \xi^i H_{ij}(X^i)\xi^j + \frac{1}{3!} a_{ijk}(X^i)\xi^i\xi^j\xi^k + \frac{1}{4!} b_{ijkl}(X^i)\xi^i\xi^j\xi^k\xi^l + ... \quad (A3)$$

From (A2), performing the ordinary formal manipulations, one obtains the on-shell (terms linear in $\xi^i$ are ignored) generating functional,

$$Z[X, 0] = Z^{(0)}[X] \quad (A4)$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left[ \int d^2 \sigma \left[ \frac{1}{3!} a_{ij}(X^i) \frac{\delta^3}{\delta J_i \delta J_j} + \frac{1}{4!} b_{ijkl}(X^i) \frac{\delta^4}{\delta J_i \delta J_j \delta J_k \delta J_l} + ... \right] \right]^n \int [d\xi^i] \exp \left\{ - \int d^2 \sigma \left[ \frac{1}{2} \xi^i H_{ij}(X^i)\xi^j + J_i \xi^i \right] \right\} \bigg|_{J=0},$$

where

$$Z^{(0)}[X] = \exp \{- \int d^2 \sigma \mathcal{L}(X^i)\} \quad (A5)$$

When dealing with one-loop computations it is enough to keep the terms quadratic in
In that case one has,
\[ Z[X,0] = \int [d\xi] \exp \left\{ \int d^2\sigma \frac{1}{2} \xi^i H_{ij}(X^i) \xi^j \right\} = \det \frac{1}{2} H(X^i), \] 
(A6)

This determinant is regulated in OR using the equations
\[ \log H = \lim_{s \to 0} \left[ -\frac{d^m}{ds^m} \frac{s^{m-1}}{m!} H^{-s} \right], \quad (m = 1, 2, 3, ...) \] 
(A7)
\[ \det H = \exp \left[ \text{tr} (\log H) \right], \] 
(A8)

with \( m = 1 \) (\( m \) is the loop order, see appendix B), and writing \( H^{-s} \) as
\[ H^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \exp (-Ht). \] 
(A9)

From the last three equations one obtains,
\[ \det H = \exp(-\zeta'(0)), \] 
(A10)

where
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{tr} \exp (-Ht). \] 
(A11)

In order to be able to compute this trace we separate \( H \) into
\[ H = H_0 + H_I, \] 
(A12)

where the trace of \( H_0 \) is known. Then, one uses the Schwinger perturbative expansion to evaluate the full trace in (A11) :
\[ \text{tr} \ e^{-Ht} = \text{tr} \left[ e^{-H_0t} + (-t)e^{-H_0t}H_I + \frac{(-t)^2}{2} \int_0^1 du e^{-(1-u)H_0t} H_1 e^{-uH_0t} H_I \right. \]
\[ \left. + \frac{(-t)^3}{3} \int_0^1 u du \int_0^1 dv e^{-(1-u)H_0t} H_1 e^{-u(1-v)H_0t} H_I e^{-uvH_0t} H_I + \ldots \right]. \] 
(A13)

After this generalities let us apply the prescription to our case. We will use a weak field expansion in powers of \( \varphi \). We have equation (A2) with the generating functional
given by $\tilde{Z}$ in (3.4). On the other hand, the operator $H$ in (A6) is the one extracted from (3.13),

$$H_{ij} = -\frac{1}{2\pi\alpha'} \delta_{ij} + M_{ij} + M_{ij}^{\alpha} \nabla_{\alpha} + M_{ij}^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta}. \quad (A14)$$

In the conformal gauge we have from (3.9),

$$\Box = e^{-\varphi} \partial^2 = (1 - \varphi + \frac{\varphi^2}{2} + \ldots ) \partial^2, \quad (A15)$$

so the most convenient way of the splitting (A12) is such that

$$H_{0ij} = -\frac{1}{2\pi\alpha'} \delta_{ij} \partial^2. \quad (A16)$$

With this choice $H_I$ has the following expansion in powers of $\varphi$,

$$H_{Iij} = \frac{1}{2\pi\alpha'} \varphi \delta_{ij} \partial^2 + M_{ij}^{(0)} + M_{ij}^{(0)} \delta^{\alpha\beta} \partial_\alpha \partial_\beta + M_{ij}^{(0)} \delta^{\alpha\beta} \delta^{\gamma\epsilon} \partial_\gamma \partial_\epsilon + \ldots \quad (A17)$$

where $M_{ij}^{(0)}$, $M_{ij}^{(0)}$ and $M_{ij}^{(0)}$ are the terms of order zero in powers of $\varphi$ in (3.14)-(3.18). Since we analytically continue the IR divergent contributions to zero the linear terms in $\varphi$ and in background fields from the expansion (A13) must come from the term quadratic in $H_I$ when one of the $H_I$ is the part linear in $\varphi$ in $\Box$, and the other $H_I$ is the $M$-part with no $\varphi$. The computation of the trace in (A13) involves an integration over the upper half-plane. This half integrations complicate the calculations so we are going to extend the integrations to the full plane. In order to do this we need to specify the value of $\varphi$ in the lower half-plane. The most convenient choice is $\varphi(\tau, \sigma) = \varphi(\tau, -\sigma), \sigma < 0$, because then the integrand corresponding to the trace in (A13) is even in $\sigma$ and so we can extend the integration region to the full plane and divide by two. Notice that this simplification is due to the fact that we are working in the conformal gauge. If one would carry the calculation for a general $\gamma_{\alpha\beta}$ this procedure would not work in all the integrations involved in the trace of (A13). One is forced to do some parts of the integration in the upper half-plane. However, when collecting all the contributions one sees that everything fits nicely in a general covariant expression. Of course, this covariant expression coincides with our result when choosing the conformal gauge.
Taking into account the considerations above, from (A6), (A10), (A11) and (A13) we have to evaluate

\[
\zeta(s) = \frac{1}{2} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \int d^2\sigma 2 \sum_{i,j} \sigma t^{1-} (1-u) \left( -\frac{1}{2} \int_0^1 du e^{-(1-u)H_0 t} \left( \frac{1}{2 \pi \alpha'} \varphi \partial^2 \delta_{ij} \right) \right) e^{-uH_0 t} \left( M_{ji}^{(0)} + M_{jia}^{(0)\delta} \partial_\beta + M_{jiab}^{(0)\delta\gamma} \partial^a \partial_\gamma \right) \sigma >
\]

(A18)

where the \(\sigma\)-integration goes over the full plane and the factor 2 in front of \(\sum_{i,j}\) is because there are two choices of \(H_I\) in (A13). In (A18) \(H_0\) is the one in (A16) without \(\delta_{ij}\). To compute (A18) we go to momentum space by defining the following Fourier transformed functions,

\[
\varphi(p) = \int \frac{d^2\sigma}{2\pi} \varphi(\sigma)e^{-i\sigma \cdot p},
\]

(A19)

\[
M_{ij}^{(0)}(p) = \int \frac{d^2\sigma}{2\pi} M_{ij}^{(0)}(\sigma)e^{-i\sigma \cdot p},
\]

(A20)

\[
M_{ij\alpha}^{(0)}(p) = \int \frac{d^2\sigma}{2\pi} M_{ij\alpha}^{(0)}(\sigma)e^{-i\sigma \cdot p},
\]

(A21)

\[
M_{ij\alpha\beta}^{(0)}(p) = \int \frac{d^2\sigma}{2\pi} M_{ij\alpha\beta}^{(0)}(\sigma)e^{-i\sigma \cdot p},
\]

(A22)

so that (A18) becomes,

\[
\zeta(s) = -\frac{1}{2} \frac{\mu^{-2s}}{\Gamma(s)} (2\pi \alpha')^{s+1} \int_0^\infty dt t^{s+1} \left[ \int_0^1 du \int \frac{d^2r \, d^2p}{(2\pi)^2} e^{-(1-u)r^2 + up^2} t \right] \varphi(r-p)^p\left( M_{ii}^{(0)}(p-r) + M_{ii}^{(0)}(p-r) i r_\alpha - M_{ii}^{(0)\alpha \beta}(p-r) r_\alpha r_\beta \right),
\]

(A23)

where a sum over \(i\) is understood and \(\mu\) is an arbitrary mass to make the equation dimensionally correct. It plays the role of the mass parameter which fixes the renormalization point. In going from (A18) to (A23) we have rescaled \(t \to 2\pi \alpha' t\) to eliminate factors involving \(2\pi \alpha'\) in the exponents. Next, we perform a shift in the variables, \(r \to r + p\), so that

\[
\zeta(s) = -\frac{1}{2} \frac{\mu^{-2s}}{\Gamma(s)} (2\pi \alpha')^{s+1} \int_0^\infty dt t^{s+1} \int_0^1 du \int \frac{d^2r \, d^2p}{(2\pi)^2} e^{-(1-u)(r+p)^2 + up^2} t \varphi(r)^p\left( M_{ii}^{(0)}(r) - M_{ii}^{(0)}(r) i r_\alpha + p_\alpha - M_{ii}^{(0)\alpha \beta}(r)(r_\alpha + p_\alpha)(r_\beta + p_\beta) \right),
\]

(A24)

i.e., all the \(p\)-dependence is taken out of the functions. To carry out the \(p\)-integrations
it is convenient to perform the shift \( p \to p - (1 - u)r \). (A24) becomes,

\[
\zeta(s) = -\frac{1}{2} \frac{\mu^{-2s}}{\Gamma(s)} (2\pi\alpha')^{s+1} \int_0^\infty dt t^{s+1} \int_0^1 du \int d^2r \varphi(r) \int d^2p \frac{p}{(2\pi)^2} e^{-|p^2 + u(1 - u)r^2|} (p - (1 - u)r)^2 \left( M_{ii}^{(0)}(-r) + M_{ii}^{(0)\alpha}(-r)i(p_\alpha + ur_\alpha) - M_{ii}^{(0)\alpha\beta}(-r)(p_\alpha + ur_\alpha)(p_\beta + ur_\beta) \right).
\]

(A25)

Now we perform the \( t \)-integral using (A9) and we get,

\[
\zeta(s) = -\frac{1}{2} \frac{\mu^{-2s}}{\Gamma(s)} (2\pi\alpha')^{s+1} \int d^2r \varphi(r) \int_0^1 du \int d^2p \frac{\mathcal{P}(p, r)}{(2\pi)^2 [p^2 + u(1 - u)r^2]^{s+2}} \Gamma(s + 2),
\]

where

\[
\mathcal{P}(r, p) = (p - (1 - u)r)^2 \left( M_{ii}^{(0)}(-r) + M_{ii}^{(0)\alpha}(-r)i(p_\alpha + ur_\alpha) - M_{ii}^{(0)\alpha\beta}(-r)(p_\alpha + ur_\alpha)(p_\beta + ur_\beta) \right).
\]

(A26)

At this point we can perform the integration in \( p \) using the standard integral used in dimensional regularization,

\[
\int \frac{d^mq}{(2\pi)^m} \frac{(q^2)^r}{(q^2 + c^2)^m} = \frac{1}{(16\pi^2)^{n/4}} \left( c^2 \right)^{s+1} \Gamma(r + \frac{n}{2}) \Gamma(m - r - \frac{n}{2}) \Gamma(\frac{n}{2}) \Gamma(m),
\]

(A27)

resulting

\[
\zeta(s) = -\frac{1}{2} \frac{\mu^{-2s}}{\Gamma(s)} (2\pi\alpha')^{s+1} \int d^2r \varphi(r) \int_0^1 du \frac{1}{4\pi} \left( (c^2)^{s+1} \Gamma(3) \Gamma(s - 1) A + (c^2)^{-s} \Gamma(s) B + (c^2)^{-s-1} \Gamma(s + 1) C \right),
\]

(A28)

where

\[
c^2 = u(1 - u)r^2,
\]

(A29)

and \( A, B \) and \( C \) are the coefficients of \( p^4, p^2 \) and \( p^0 \) in \( \mathcal{P}(p, r) \):

\[
A = -\frac{1}{2} \delta_{\alpha\beta} M_{ii}^{(0)\alpha\beta}(-r),
\]

\[
B = M_{ii}^{(0)}(-r) - i M_{ii}^{(0)\alpha}(-r)(1 - 2u)r_\alpha - M_{ii}^{(0)\alpha\beta}(-r) \left( \frac{1}{2} \delta_{\alpha\beta}(1 - u)^2 r^2 + r_\alpha r_\beta(u^2 - 2u(1 - u)) \right),
\]

\[
C = (1 - u)^2 r^2 \left( M_{ii}^{(0)}(-r) + u M_{ii}^{(0)\alpha}(-r)i r_\alpha - u^2 M_{ii}^{(0)\alpha\beta}(-r)r_\alpha r_\beta \right).
\]

(A30)
The $u$-integration is carried out using

$$
\int_0^1 du u^{r-1}(1-u)^{s-1} = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)},
$$

(A32)

and finally we obtain

$$
\zeta(s) = -\frac{1}{2} \mu^{-2s}\frac{1}{4\pi} (2\pi\alpha')^{s+1} s \int d^2r \varphi(r) \left[ M^{(0)ii}(r) + \frac{i}{2} r_\alpha M^{(0)\alpha\alpha}(r) \right.

+ \left. \frac{1}{3} \left( \frac{1}{4} r^2 \delta_{\alpha\beta} - r_\alpha r_\beta \right) M^{(0)\alpha\beta}(r) \right] + O(s^2).
$$

(A33)

Notice that the $\mu$-dependence drops out of the result since the effective action we are after involves the $s$-derivative of $\zeta(s)$ evaluated at $s = 0$. The cancellation of the leading term in the expansion of the $\Gamma$-functions that arise from the $u$-integration is highly non-trivial and provides an excellent test of the calculation. Going back to $\sigma$-space we get for the effective action $\Gamma (\tilde{Z} = \exp(-\Gamma))$, after using (A10),

$$
\Gamma^{(1)} = \frac{\alpha'}{2} \int_\mathcal{M} d^2\sigma \varphi(\sigma) \left[ M^{(0)ii}(\sigma) - \frac{1}{2} \partial_\alpha M^{(0)\alpha\alpha}(\sigma) - \frac{1}{3} \left( \frac{1}{4} \partial^2 \delta_{\alpha\beta} - \partial_\alpha \partial_\beta \right) M^{(0)\alpha\beta}(\sigma) \right],
$$

(A34)

which is the final result in this appendix. In this expression the superindex (1) indicates that it corresponds to the part of the effective action which contains all loop corrections at order linear in $\varphi$ and in the background fields.
In this appendix we describe the computations leading to (5.3) and we prove that there are not two-loop contributions to the structure we have considered in sect. 5.

At quadratic order in $\varphi$ and using three $H_I$ in the expansion (A13) we have, after taking into account (A11), (A13), (3.13) and (3.19) that the contribution to the effective action is $-\zeta'(0)$ where,

$$\zeta(s) = \frac{1}{2} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \int d^2 \sigma 3 \sum_{i,j,k} <\sigma|(-t)^3 \int_0^1 ududv -(1-u)H_0 t \left( \frac{1}{2\pi} \varphi \delta_{ij} \partial^2 \right) e^{-u(1-v)H_0 t} \left( M_{ki}^{(0)} + M_{ki}^{(0)\alpha} \partial_\alpha + M_{ki}^{(0)\alpha\beta} \partial_\alpha \partial_\beta \right) |\sigma> .$$

(B1)

In this equation the factor $\frac{1}{2}$ in front comes because we are integrating over the full plane and the factor 3 right before the summation symbol because there are three different arrangements for the $H_I$ and all give the same contribution. In momentum space, after using (A19)-(A22), (B1) becomes,

$$\zeta(s) = -\frac{1}{2} \frac{\mu^{-2s}}{\Gamma(s)} (2\pi)^{s+1} \int_0^\infty dt t^{s+2} \int_0^1 ududv \int \frac{d^2 p d^2 q d^2 r}{(2\pi)^3} e^{-[(1-u)p^2 + u(1-v)q^2 + uv r^2] t} \varphi(p-q)\varphi(q-r) \left( M_{ii}^{(0)} (r-p) + iM_{ii}^{(0)\alpha} (r-p) p_\alpha - M_{ii}^{(0)\alpha\beta} (r-p) p_\alpha p_\beta \right).$$

(B2)

where we have introduced the arbitrary mass parameter $\mu$ and we have rescaled $t \rightarrow 2\pi t$ (we set $\alpha' = 1$ all along the appendix). Performing the following series of shifts in momenta, $p \rightarrow p + q$, $q \rightarrow q + r$, and $r \rightarrow r - [(1-u)p + (1-uv)q]$, we find

$$\zeta(s) = -\frac{1}{2} \frac{\mu^{-2s}}{\Gamma(s)} (2\pi)^{s+1} \int_0^\infty dt t^{s+2} \int_0^1 ududv \int \frac{d^2 p d^2 q d^2 r}{(2\pi)^3} \left( \varphi(p)\varphi(q) \exp \left\{ - [r^2 + p^2 u(1-u) + q^2 uv(1-uv)] + 2p \cdot q (1-u)uv \right \} \right.$$

$$\left. [r - (1-u)p + uvq]^2 [r - (1-u)p - (1-uv)q]^2 \left( M_{ii}^{(0)} (-p - q) + iM_{ii}^{(0)\alpha} (-p - q) (r + up + uvq)_\alpha \right.$$

$$\left. - M_{ii}^{(0)\alpha\beta} (-p - q) (r + up + uvq)_\alpha (r + up + uvq)_\beta \right). \right.$$  

(B3)

This expression is now ready to carry out the $t$ and $r$ integrations. So far we have kept the complete contribution of this type. However, since we are interested in a particular
structure, namely, the one containing \( p^2 \), \( q^2 \) or \( p \cdot q \) in momentum space, we will keep from now on the part of (B3) proportional to \( r_\alpha r_\beta \). This simplify matters since after performing the \( t \)-integration, we get

\[
\zeta(s) = \frac{1}{4} \mu^{-2s} (2\pi)^{s+1} \Gamma(s + 3) \int_0^1 ududv \int \frac{d^2p d^2q d^2r}{(2\pi)^3} \left[ \varphi(p) \varphi(q) \right. \\
\left. \frac{r^6 + r^4 \mathcal{P}(p, q; u, v) + r^2 \mathcal{Q}(p, q; u, v)}{r^2 + C(p, q; u, v)} \right]^{s+3},
\]

where

\[
\mathcal{C}(p, q; u, v) = p^2 u (1 - u) + q^2 uv (1 - uv) + 2p \cdot q uv (1 - u), \tag{B5}
\]
\[
\mathcal{P}(p, q; u, v) = (1 - 3(1 - uv)uv)q^2 + 3(1 - u)^2 p^2 \\
+ 3(1 - u)(1 - 2uv)p \cdot q, \tag{B6}
\]
\[
\mathcal{Q}(p, q; u, v) = q^4 (1 - uv)^2 (uv)^2 - 2q^2 p \cdot q (1 - u)(1 - uv)uv (1 - 2uv) \\
+ q^2 p^2 (1 - u)^2 (uv + 2(1 - uv)^2) - 4(p \cdot q)^2 (1 - u)^2 (1 - uv)uv \\
+ 2p^2 p \cdot q (1 - u)^3 (1 - 2uv) + p^4 (1 - u)^4. \tag{B7}
\]

The \( r \)-integration in (B4) is performed using (A28),

\[
\zeta(s) = \frac{1}{4} \mu^{-2s} (2\pi)^{s+1} \Gamma(s + 3) \int ududv \left[ \mathcal{C}(p, q; u, v)^{1-s} + 2\mathcal{C}(p, q; u, v)^{-s} \mathcal{P}(p, q; u, v) \\
+ s\mathcal{C}(p, q; u, v)^{-s-1} \mathcal{Q}(p, q; u, v) \right]. \tag{B8}
\]

We are interested in the expression of \( I(p, q) \) up to linear terms in \( s \) and we expect the independent term to vanish (otherwise we would get a \( \mu \)-dependence in \( \zeta'(0) \)). From (B5)-(B7) one can observe that the integrations of \( \mathcal{C}(p, q; u, v)^{1-s} \), \( \mathcal{C}(p, q; u, v)^{-s} \mathcal{P}(p, q; u, v) \) and \( \mathcal{C}(p, q; u, v)^{-s-1} \mathcal{Q}(p, q; u, v) \) are not divergent in the limit
s \to 0. Therefore we are allowed to expand the integrand in powers of \( s \),

\[
I(p, q) = \int_0^1 \int_0^1 ududv \left[ -6C(p, q; u, v) + \mathcal{P}(p, q; u, v) \right] \\
+ s \int_0^1 \int_0^1 ududv \left[ 6C(p, q; u, v) (-1 + \log C(p, q; u, v)) \right. \\
\left. - 2\mathcal{P}(p, q; u, v) \log C(p, q; u, v) \right] + O(s^2).
\]

\( \tag{B10} \)

The integration of the first term is elementary and one can verify that indeed it is zero confirming the fact that \( \zeta'(0) \) is independent of \( \mu \). On the other hand, looking at (B5)-(B7) the integration of the second term seems rather cumbersome. However, based on the arguments discussed above one expects a simple answer, namely, one expects it to be \( \alpha p^2 + \beta q^2 + \gamma p \cdot q \) where \( \alpha, \beta \) and \( \gamma \) are numbers. If this were true (in fact it is shown below that it is) one could easily obtain \( \alpha, \beta \) and \( \gamma \) considering \( I(p, q) \) for the cases, \( p^2 = q^2 = 0, p^2 = p \cdot q = 0, \) and \( q^2 = p \cdot q = 0 \). In those cases the calculation of \( I(p, q) \) can be carried out analytically. If \( p^2 = q^2 = 0 \),

\[
I_1(p, q) = (2p \cdot q)^{1-s} \int_0^1 \int_0^1 ududv \left[ 6 \frac{(1-u)uv}{s-1}^{(1-s)} - s \frac{(1-u)uv}{s-1}^{(1-s)} (1-u)(1-2uv) \right.
\left. + 3 (1-u)uv^{-s} (1-u)(1-2uv) \right]
\]

\[
= - \frac{1}{4} s p \cdot q + O(s^2),
\]

\( \tag{B11} \)

after having used (A32). Similarly, for the other two cases,

\[
I_2(p, q) = - \frac{1}{12} s q^2 + O(s^2), \quad (B12)
\]

\[
I_3(p, q) = - \frac{1}{4} s p^2 + O(s^2), \quad (B13)
\]

Our formal arguments lead us to conjecture that the full answer is just the sum of (B11)-(B13),

\[
I(p, q) = - \frac{1}{4} (p^2 + \frac{1}{3} q^2 + p \cdot q) + O(s^2). \quad (B14)
\]

We have evaluated the second part of (B10) numerically and it turns out that indeed (B14) is the complete answer. It is extraordinary that an integration as involved as
the one in (B10) has such a simple result. Similar patterns should occur when dealing with other structures.

Going back to (B8) we finally have:

\[ \zeta(s) = s \frac{1}{32} \int \frac{d^2 p d^2 q}{2\pi} \varphi(p) \varphi(q) M^{(0)\alpha}_{ii\alpha} (-p - q) (p^2 + \frac{1}{3} q^2 + p \cdot q) + O(s^2), \quad \text{(B15)} \]

which, in configuration space gives the following contribution to the effective action:

\[ \Gamma^{(2)}_I = \frac{1}{16} \int_{\mathcal{M}} d^2\sigma \left( \frac{4}{3} \varphi(\sigma) \partial^2 \varphi(\sigma) + \partial^\alpha \varphi(\sigma) \partial_\alpha \varphi(\sigma) \right) M^{(0)\beta}_{ii\beta}(\sigma). \quad \text{(B16)} \]

In the rest of this appendix we will show that there are not contributions from two loops to the structure \( \partial_\sigma \varphi \partial_\sigma \varphi \) selected in sect. 5. From (A4) and (A6) the form of the generating functional becomes,

\[ Z[X, 0] = Z^{(0)}[X] \text{det}^{-\frac{1}{2}} H(X) \]

\[ \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \int d^2\sigma \left[ \frac{1}{3!} a_{ijk}(X) \delta^3 \frac{\delta^3}{\delta J_i \delta J_j \delta J_k} + \frac{1}{4!} b_{ijkl}(X) \delta^4 \frac{\delta^4}{\delta J_i \delta J_j \delta J_k \delta J_l} + \ldots \right] \right]^n \exp\left\{ - \int d^2\sigma \left[ \frac{1}{2} J^i H^{-1}_{ij} J^j \right] \right\} \right|_{J=0}. \quad \text{(B17)} \]

Higher-loop contributions are originated from terms in the expansion (3.5) with higher powers of \( \xi^i \). At order linear in the background fields and quadratic in \( \varphi \) there are only two-loop contributions. All others contain tadpole-like diagrams which are continued to zero. The terms in the expansion (3.5) which are relevant contain four \( \xi^i \)'s. For example, if we concentrate our attention in the first massive mode one finds in (3.5) a term of the form

\[ S^{(4)}(\Phi; \varphi, X, \xi) = \ldots + \int_{\partial\mathcal{M}} e d\tau \frac{1}{2} (\partial_i \partial_j A_{kl} - \partial_i \partial_j \partial_k B_l) \frac{1}{e d\tau} \xi^i \frac{1}{e d\tau} \xi^k \xi^j \xi^i \]

\[ = \ldots + \int_{\mathcal{M}} d^2\sigma e^{\varphi} L^{\alpha\beta}_{ijkl} \nabla_a \xi^i \nabla_\beta \xi^k \xi^j \xi^i + \ldots \quad \text{(B18)} \]

where, as usual, we have extended the integration region to the full upper half-plane.
after defining
\[ L_{ijkl}^{\alpha\beta} = \begin{cases} \left[ \frac{1}{2} \partial_i \partial_j A_{kl} - \frac{1}{2} \partial_i \partial_j B_i \right] (\tau) e^{\frac{\tau}{2}} \delta(\sigma), & (\alpha, \beta) = (0, 0) ; \\ 0, & \text{otherwise}. \end{cases} \]  
\[ (B19) \]

Performing the functional variation in (B17) and using (B18), we obtain,
\[ Z \approx \int d^2 \sigma L_{ijkl}^{\alpha\beta} \left( \delta_{ij} \delta_{kl} < \sigma | H^{-1} | \sigma > - \frac{1}{2} \delta_{ij} \delta_{kl} \right) \left( \nabla_{\alpha} \nabla_{\beta} H^{-1} | \sigma > - \frac{1}{2} \delta_{ij} \delta_{kl} \right). \]  
\[ (B20) \]

where \( H = H_0 + H_I \) (\( H_0 \) is the one in (A16) without \( \delta_{ij} \), and \( H_I \) is the first term of (A17) without \( \delta_{ij} \)). We see from this expression that the two-loop calculation factorizes in a product of two one-loop computations. The prescription of OR to regulate strings of inverses of operators at \( m \)-loop order is based in (A7) and (A9),
\[ A_1^{-1} A_2^{-1} \ldots A_p^{-1} = \lim_{s \to 0} \left[ \frac{d^m}{ds^m} \left[ \frac{s^m}{m!} A_1^{-s-1} A_2^{-s-1} \ldots A_p^{-s-1} \right] \right], \]  
\[ (B21) \]

with
\[ A^{-s-1} = \frac{1}{\Gamma(s+1)} \int_0^\infty dt t^s e^{-At}. \]  
\[ (B22) \]

The computation of (B20) is similar to the ones encountered at one loop after using another perturbative expansion due to Schwinger:
\[ e^{-(A_0 + A_I) t} = e^{-(A_0) t} + (-t) \int_0^1 du e^{-(1-u)A_0 t} A_I e^{-u A_0 t} + \ldots \]  
\[ (B23) \]

We find that (B20) has the form,
\[ Z \approx \int \frac{dp dq}{2\pi} \varphi(p) \varphi(q) L_{ijkl}^{\alpha\beta} (-p - q) \left( \delta^{ij} \delta^{kl} \left( \frac{1}{3} q_\alpha q_\beta - \frac{1}{12} \delta_{\alpha\beta} q^2 \right) - \frac{1}{2} \delta^{ik} \delta^{jl} p_\alpha q_\beta \right). \]  
\[ (B24) \]

Certainly, this expression will never contribute to the structure selected in sect. 5. Remember from (B15) and (B16) that one needs a \( p \cdot q \) in momentum space to generate such structure.
It is worth noting that at linear order in the background fields all the contributions coming from higher loops factorize into products of one loop-calculations. Furthermore, the number of loops which are relevant is the same as the number of powers of $\varphi$ one is considering in the effective action. From these facts it is simple to argue that at linear order in background fields there are not higher-loop contributions to the structure selected in sect. 5. Because of the factorization it is not possible to generate terms in momentum containing $p \cdot q$, which, as shown in (B15) and (B16), is the form connected to the structure selected in sect. 5.
REFERENCES