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OPERATOR FORMALISM FOR CHERN-SIMONS THEORIES

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ABSTRACT

The operator formalism for Chern-Simons theories with gauge group G and parameter k ($G = U(1), SU(2)$) on an arbitrary oriented compact three-dimensional manifold is constructed. The states of the Hilbert space are obtained in a wave-functional representation which corresponds to a generalized form of the exponential of a Wess-Zumino-Witten action. It is shown explicitly that the states of a basis of the Hilbert space are in one-to-one correspondence with the characters of a Wess-Zumino-Witten model with gauge group G at level k and have their same properties under modular transformations. In addition, it is also shown that the Wilson line operators with gauge field in some distinguished representations act as creation operators in the Hilbert space and verify the fusion algebra of the corresponding conformal field theory.

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1. Introduction. Recently, Witt
dimensional topological field theories
and conformal field theories in two d
simply connected compact Lie group
Hilbert space obtained upon quantiz
surface can be interpreted as the spac
relation has been made somehow m
extended to any compact Lie group.
has been presented in [3] for the case
construction of the Hilbert space is
the characters of a two-dimensional *c*
analysis, as well as the ones recently
quantization of Chern-Simons theorie
where Σ is a Riemann surface with
the connection between Chern-Simo
field theories by means of the constr
oriented three-dimensional surfaces w
in spirit to the one in string theory
defined as wave functionals via a fun
the symmetry of the theory these st
such a Hilbert space is obtained for
The resulting formulation provides a
For the case in which no Wilson line
of the Hilbert space of states can b
conformal field theory, *i.e.*, their elen
properties under modular transform
to-one correspondence with the elen
like the Verlinde operators [6] of the
letter we will give a brief report of
the details of our work will be preser

2. Operator formalism: Abelian case. Let us consider an oriented compact three-dimensional surface without boundary M and an $U(1)$ bundle E (which may well be trivial) endowed with a connection A_μ (which can be viewed as a one-form). The Feynman path integral of the corresponding Chern-Simons theory is defined as

$$Z(M)_k = \int [DA_\mu] \mathrm{e}^{ikS(A_\mu)}, \tag{1}$$

with

$$S(A_\mu) = \frac{1}{2\pi} \int_M A \wedge dA, \tag{2}$$

where $[DA_\mu]$ represents Feynman's path integral over gauge orbits and k is an arbitrary integer. The reason why we must integrate over gauge orbits, *i.e.*, under all equivalent classes of connections modulo gauge transformations, is that when k is integer the argument of the functional integral in (1) is invariant under arbitrary gauge transformations $A_\mu \to A_\mu + g^{-1}\partial_\mu g$ where g is an arbitrary continuous map $g : M \to U(1)$ [8]. This gauge invariance plays a fundamental role in our construction.

All three-dimensional manifolds of the type considered here admit a Heegaard splitting [9]. This means that we may cut M along a Riemann surface Σ of genus g in such a way that $M = M_1 \cup M_2$ being M_1 and M_2 homeomorphic to a solid ball with g handles. The joining of M_1 and M_2 to build M is carried out by identifying their surfaces ∂M_1 and ∂M_2 via an homeomorphism. Of course, a given M admits many different splittings. To fix ideas let us consider for example $g = 1$. Let M_1 and M_2 be two identical solid tori with modular parameter τ. If they are joined by identifying the points of their surfaces the manifold $S^2 \times S^1$ is constructed. However, if before the identification a modular transformation which transforms $\tau \to -1/\tau$ is made, the resulting manifold is S^3. The manifold S^3 admits also a $g = 0$ Heegaard splitting just cutting it along S^2.

The operator formalism which we are about to construct will consist of states defined as functional integrals over configurations of gauge fields on either M_1 or

M_2 after cutting M via a Heegaard sp
which are the arguments of the corre
must correspond to field configurati
$Z(M)_k$ must be evaluated by a suita
natural way of making this choice is
of the canonical quantization of the
[3]. We define:

$$\Psi(A_{\bar z}) = \int [DA_\mu]_{M_1} \mathrm{ex}$$

where $[DA_\mu]_{M_1}$ represents the Feynn
such that $A_{\bar z}$ is fixed at ∂M_1, and
$\frac{1}{2}(A_1 + iA_2)$, A_0 being in the directi
real local coordinates on the Riemann
∂M_1 and choosing local coordinates
the theory is topological, one would
is independent of this choice. Howe
conformal anomaly. Let us discuss
general analysis in [1] the state defin
bundle on moduli space. This vect
is only projectively flat because of
the scale of the metric in (3) corres
definition (3) we have added to the
is conformal invariant and so we do
point. However, the dependence on
the functional integral and it will ap

Similarly, we define the state co

$$\Phi(A_z) = \int [DA_\mu]_{M_2} \mathrm{ex}$$

so we may write $Z(M)_k$ as

$$Z(M)_k = \int [DA_z DA_{\bar z}] \exp(\frac{2k}{\pi} \int_{\Sigma} d^2\sigma\, A_z A_{\bar z}) \Phi(A_z) \Psi(A_{\bar z}), \qquad (5)$$

where $\Sigma = \partial M_1$. Let us compare this expression with the one resulting from a canonical quantization in an axial-time gauge in the holomorphic representation [3]. On one hand the exponential factors are the same. On the other hand, in the situation where a Heegaard splitting is such that ∂M_1 and ∂M_2 are identified without previously making any homeomorphism, $\Phi(A_z) = \overline{\Psi(A_{\bar z})}$, as follows from (3) and (4). We have defined (3) and (4) based on our intuition from the canonical quantization of the theory. Alternatively, we could think that (3) and (4) are defined in that way because then they are rather simple to compute, and then from (5) we could read the quantization relations of the theory. For a linear theory, such as the Abelian case we are treating in this section, there is no advantage in taking one point of view or the other. However, for the non-Abelian case non-linearities play a fundamental role and the second point of view is much more fruitful. As we will discuss in the next section a suitable definition of the states allows us to determine the Hilbert space and this in turn permits to read the "full" commutation relations from an expression similar to (5).

Our next task is to determine $\Psi(A_{\bar z})$ in (3) exploiting the symmetry present in the theory. It is rather straightforward to verify that the functional integral in (3) has its extremal at field configurations such that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$ on M_1, and that under a gauge transformation $A_\mu \to A_\mu + g^{-1}\partial_\mu g$, where g is a continuous mapping $g : M_1 \to U(1)$, it transforms as

$$\Psi(A_{\bar z}) \to \mathrm{e}^{-2k(\gamma(g) + \frac{1}{\pi}\int_{\partial M_1} d^2\sigma\, A_{\bar z}\partial_z g g^{-1})} \Psi(A_{\bar z}), \qquad (6)$$

where

$$\gamma(g) = \frac{1}{2\pi} \int_{\partial M_1} d^2\sigma\, g^{-1}\partial_z g g^{-1} \partial_{\bar z} g. \qquad (7)$$

The first consequence of (6) is that the state defined by (3) satisfies the Gauss law $F_{z\bar z}\Psi(A_{\bar z}) = 0$. This is easily pr
(6) and the commutation relations st
(5).

The mappings g on M_1 are clas
they wrap around non-contractible l
closed cycles on ∂M_1 which generat
and let us assume that our solid ba
cycles are contractible. Dual to these
defined on ∂M_1 which allow us to de
with the help of the period matrix Ω
differentials satisfy $\int_{\alpha_i} \omega_j = \delta_{ij}$ an
$\mathrm{Im}\Omega_{ij}$.

Considering the analysis in [10]
the gauge fields: $A_{\bar z} = (u_a u)^{-1}\partial_{\bar z}(u_a$
to the identity map $u : \partial M_1 \to U(1)$
$u_a = \exp(\pi \int^{\bar z} \overline{\omega(z)} (\mathrm{Im}\Omega)^{-1} a - \pi \bar a (\mathrm{I}$
valued on ∂M_1 and that our param
holonomy phases around α_i and β_i c
In this parametrization gauge transf
to (6) has the form

$$\Psi(A_{\bar z}$$

where $\mathrm{e}^{-\gamma_{2k}(u_a)}$ is defined as a func
satisfies

$$\mathrm{e}^{-\gamma_{2k}(u_a g)} = \mathrm{e}^{-}$$

being $< u_a, g >= \frac{1}{\pi}\int d^2\sigma\, u_a^{-1}\partial_{\bar z} u_a$
map connected to the identity $< u_a$
(8) factorizes in u_a and u. In this c
$\mathrm{e}^{-\gamma_{2k}(u_a)}$. However, if g is a map wh

α_j, *i.e.*, $g = \exp\big(-\pi(n+m\bar{\Omega})(\mathrm{Im}\Omega)^{-1}\int^z \omega(z) + \pi\int^{\bar{z}}\overline{\omega(z)}(\mathrm{Im}\Omega)^{-1}(n+\Omega m)\big)$ we obtain the condition:

$$\mathrm{e}^{-\gamma_{2k}(u_{a+n+\Omega m})} = \mathrm{e}^{-\gamma_{2k}(u_a)}\mathrm{e}^{\pi k(2(n+m\bar{\Omega})(\mathrm{Im}\Omega)^{-1}a+(n+m\bar{\Omega})(\mathrm{Im}\Omega)^{-1}(n+\Omega m))}, \quad n,m\in Z^g. \tag{10}$$

The most general entire function of a which verifies (10) is a linear combination of

$$\psi_p(a) = \xi\mathrm{e}^{\pi k a(\mathrm{Im}\Omega)^{-1}a}\vartheta\begin{bmatrix}\frac{p}{2k}\\ 0\end{bmatrix}(2ka|2k\Omega), \quad p\in(Z_{2k})^g, \tag{11}$$

where ϑ is the Jacobi theta function with characteristics [11], and ξ is a constant (independent of a).

Using symmetry arguments we have found $(2k)^g$ functions in our search for the form of the functional integral (3). This non-uniqueness is expected. The non-trivial gauge invariant operators of the theory are Wilson lines around non-contractible cycles. If we have had a Wilson line inserted in the solid ball with g handles in the definition (3) all our arguments based on the symmetries of the theory would be still valid. Therefore, one does not expect to find a unique state but the full set of states of the Hilbert space. To find a basis we must study the orthogonality relations of the states we have obtained. Let us define

$$\Psi_p(A_{\bar{z}}) = \mathrm{e}^{-2k\gamma(u)}\psi_p(a), \tag{12}$$

and let us consider the case in which $M = S^2\times S^1$ and M_1 is the solid torus. From (5) we have the inner product

$$(\Psi_q,\Psi_p) = \int[DA_zDA_{\bar{z}}]\mathrm{e}^{\frac{2k}{\pi}\int d^2\sigma\, A_zA_{\bar{z}}}\overline{\Psi_q(A_{\bar{z}})}\Psi_p(A_{\bar{z}}). \tag{13}$$

The measure in (13) is defined by $||\delta A|| = \int d^2\sigma\,\delta A_z\delta A_{\bar{z}}$. In computing this inner product one finds determinants of operators which must be regulated. Here is where the conformal anomaly enters in our formulation. Different choices of the

scale of the metric lead to different
to lead to the simplest formulation
$\int_{\partial M_1} d^2\sigma\sqrt{g} = 2\mathrm{Im}\tau$, where τ deno
torus. One finds from (13), (11) and

(Ψ_q,Ψ_p)

where $\eta(\tau)$ is the Dedekind η functio
orthogonal and so they constitute a
made orthonormal by defining $\xi = \eta$
[1] to show that it is in $S^2\times S^1$ whe

From (11), (12) and (14) we hav
Hilbert space of the theory consists

$\Psi_p(A_{\bar{z}}) = \mathrm{e}^{-2k\gamma(u)}\mathrm{e}^{\pi k a(\mathrm{Im}\tau)}$

These functionals transform as the s
ular group of the torus. Using stand
Dedekind η function one finds that
modular group $S: a\to a/\tau, \tau\to -1/$

$\Psi_p|_S =$

$\Psi_p|_T = \mathrm{e}^2$

From the form of the T transforma
$h_p = \frac{p^2}{4k}$ and the central charge $c = 1$

As we mentioned above, the sta
respond to Feynman's path integral
been inserted. Now we will comput

and unlinked Wilson line of charge n, $\phi_n = \exp(-n\int_\beta A)$. Since it is unknotted and unlinked we may translate it to the surface of the solid torus by a continuous deformation. As the system has zero Hamiltonian, this operation leaves invariant the value of the path integral. Once on the surface, using the holomorphic representation dictated by (5), *i.e.*, $\bar{a} = \frac{\mathrm{Im}\tau}{2k\pi}\frac{\partial}{\partial a}$, the Wilson line can be written in operator form and we have

$$\phi_n\Psi_p = \exp(-n\int_\beta A)\Psi_p = \mathrm{e}^{-\frac{\pi n^2}{4k}\bar{\tau}(\mathrm{Im}\tau)^{-1}\tau - \pi n\bar{\tau}(\mathrm{Im}\tau)^{-1}a}\mathrm{e}^{\frac{n\tau}{2k}\frac{\partial}{\partial a}}\Psi_p = \Psi_{p+n}. \quad (17)$$

In view of this result the most natural interpretation is to associate the state Ψ_0 to the case in which there are no Wilson lines inside the solid torus and the state Ψ_p to the case in which there is an unknotted Wilson line of charge p. We observe that there are only k distinguished charges. The states (15) are in one-to-one correspondence with the characters of a rational Gaussian model [12] and share their same properties under modular transformations. The operators ϕ_p may be identified with the primary fields of those models. Certainly, if one considers two unknotted unlinked Wilson lines with charges p and q, the effect of acting on the states (15) is commutative and one has the same effect as if one considers only a Wilson line with charge $p+q$. In other words, they satisfy the fusion rule $\phi_p \times \phi_q = \phi_{p+q}$.

Once the states (15) are constructed and the unitary representation of the modular group (16) is worked out it is rather simple to compute the partition function (5). As an example let us calculate $Z(S^3)_k$. Since S^3 has a Heegaard splitting which involves a homeomorphism corresponding to an S modular transformation, from (14), (15) and (16) one finds $Z(S^3)_k = 1/\sqrt{2k}$.

3. Operator formalism: non-Abelian case. We will consider a non-Abelian Chern-Simons theory for the concrete case of the group $SU(2)$. However, all our arguments can be generalized to an arbitrary Lie group. We will be dealing with

a Feynman path integral like the one

$$S(A_\mu) = \frac{1}{4\pi}\int_M \mathrm{T}$$

where 'Tr' denotes the trace in the f
Abelian case $\exp(ikS)$ is invariant u
$g^{-1}\partial_\mu g$ where g is an arbitrary conti

Let us consider first the case in w
splitting such that M_1 and M_2 are tw
boundary. Similarly to the Abelian c

$$\Psi(A_{\bar{z}}) = \int [DA_\mu]_{M_1} \exp\big($$

and accordingly (see (4)) for the sta
considered, $\Phi(A_z) = \overline{\Psi(A_{\bar{z}})}$. In the
one subtlety related to the conforma
until the computation of the norm o
an additional subtlety which is fund
path integral (19) we must perform a
in which the radial component of A
then reduces to A_μ configurations o
is still a residual gauge invariance in
Let us parametrize these components
the boundary. We choose a measure
and by continuity we must have the

$$[DA_zDA_{\bar{z}}] = \mathrm{e}^{c}$$

where $A_{\bar{z}} = u^{-1}\partial_{\bar{z}}u$ and $A_z = \bar{u}^{-1}\partial$
$u^{-1} = \bar{u}^\dagger$), c_{v} is the quadratic Casimi