6 LECTURES ON QFT, RG AND SUSY

TIMOTHY J. HOLLOWOOD

Department of Physics, Swansea University, Swansea, SA2 8PP, UK t.hollowood@swansea.ac.uk

Lectures given at the British Universities Summer School in Theoretical Elementary Particle Physics (BUSSTEPP) Liverpool University August 2009 and Cambridge University August 2008.

Abstract

An introduction to the theory of the renormalization group in the context of quantum field theories of relevance to particle physics is presented in the form of 6 lectures delivered to the British Universities Summer School in Theoretical Elementary Particle Physics (BUSSTEPP). Emphasis is placed on gaining a physical understand of the running of the couplings and the Wilsonian version of the renormalization group is related to conventional perturbative calculations with dimensional regularization and minimal subtraction. An introduction is given to some of the remarkable renormalization group properties of supersymmetric theories. The purpose of these lectures is to introduce a powerful way to think about Quantum Field Theories (QFTs). This conceptual framework is Wilson's version of the Renormalization Group (RG). The only pre-requisites are a basic understanding of QFTs along the lines of a standard introductory QFT course: the Lagrangian formalism, propagator, Feynman rules, the path integral formulation, *etc*.

It turns out that supersymmetric theories are a wonderful arena for discussing RG because there are many things that one can prove exactly. For this reason the last two lectures will provide a very basic description of some of the extraordinary features that SUSY theories have in regard to the RG. The discussion of SUSY will necessarily be very rudimentary.

I have not included any references in these notes. However, sources which I have found particularly useful are:

(1) The idea of the renormalization group goes back quite a long way. In this course we have in mind the version of the renormalization group due to Wilson and one can do no better than consult the Physics Reports (Volume 12, Number 2, 1974) by Wilson and Kogut.

(2) Peskin and Schroeder's textbook on QFT "Quantum Field Theory" is probably the most useful and will fill in many of the gaps I have left. I also like Zinn-Justin's huge "Quantum Field Theory and Critical Phenomena".

(3) Weinberg's notes on "Critical Phenomena for Field Theorists" are very useful for explaining how many of the ideas of the renormalization group came from statistical physics and are available in scanned form at http://ccdb4fs.kek.jp/cgi-bin/img/allpdf?197610218.

(4) Manohar's notes on "Effective Field Theories" hep-ph/9606222 have an excellent discussion of the delicate issue of decoupling in momentum-dependent and -independent RG schemes.

(5) Strassler's "An Unorthodox Introduction to Supersymmetric Gauge Theory" are not only unorthodox but excellent because they tackle many issues that are not covered in conventional SUSY texts.

(6) I will not use superspace when discussing SUSY, but a standard source for this booking-keeping device is the book "Supersymmetry and Supergravity" by Wess and Bagger.

(7) Our discussion of the beta function of SUSY gauge theories has been heavily influenced by 2 very insightful papers by Arkani-Hamed and Murayama, hep-th/9705189 and hep-th/9707133 which clarify (at least for me) many of the confusing issues regarding renormalization in SUSY gauge theories.

Naturally, a series of only 6 lectures can only scratch the surface of this subject and I have had to leave out lots of things and simply the discussion. Some additional information appears in the form of a set of notes at the end of each lecture.

I would like to thank the organizers of BUSSTEPP, Jonathan Evans in Cambridge 2008 and Ian Jack in Liverpool 2009, for providing excellently run summer schools that enabled me to develop my idea to teach QFT with the renormalization group as the central pillar.

1 The Concept of the Renormalization Group

The key idea of the renormalization group results from comparing phenomena at different length/energy scales. A QFT is defined by an action functional of the fields $S[\phi; g_i]$ which depends a priori on an infinite number of parameters: the coupling constants in a general sense so including mass parameters, *etc.*. The set of couplings $\{g_i\}$ can be thought of as a set of coordinates on *theory space*. The functional integral takes the form

$$\mathscr{Z} = \int [d\phi] \, e^{iS[\phi;g_i]} \tag{1.1}$$

and so, as well as the action, we have to define the measure $\int [d\phi]$. This is a very tricky issue since a classical field has an infinite number of degrees-of-freedom and it is by no means a trivial matter to integrate over such an infinite set of variables. In perturbation theory a symptom of the difficulties in defining the measure shows up as the divergences that occur in loop integrals. These UV divergences occur when the momenta on internal lines become large so they are intimately bound up with the fact that the field has an infinite number of degrees-of-freedom and can fluctuate on all energy scales. Of course there may also be IR divergences, however these are not as conceptually serious as the UV ones: in reality in a real experiment one is working in some finite region of spacetime and this provides a natural IR cut-off.

At least initially, in order to make sense of $\int [d\phi]$, we have to implement some UV cut-off procedure in order to properly define the measure, or equivalently, in perturbation theory regulate the infinities that occur in loop diagrams. As we have said above, these UV, high energy divergences occur because the fields can fluctuate at arbitrarily small distances and in order to regulate the theory we have to somehow suppress these high energy modes. Whatever way this is done inevitably introduces a new energy scale μ , the *cut-off*, into the theory.

Cut-offs or Regulators

There are many ways of introducing a cut-off, or regulator, into a QFT. For example, one can define the theory on a spatial lattice (after Wick rotation to Euclidean space). In this case μ^{-1} is the physical lattice spacing. Or one can suppress the high momentum modes by modifying the action or the measure. Or in perturbation theory one can analytically continue the spacetime dimension, a procedure known as dimensional regularization.

Suppose we have some physical quantity $\mathscr{F}(g_i; \ell)_{\mu}$ which can depend in general on a (or possibly several) length scale ℓ (or equivalently an energy scale $1/\ell$). The theory of RG postulates that one can change the cut-off of the theory in such a way that the physics on energy scales $< \mu$ remains constant. In order that this is possible the couplings must change with μ . This idea can be summed up in the

RG equation:¹

$$\mathscr{F}(g_i(\mu);\ell)_{\mu} = \mathscr{F}(g_i(\mu');\ell)_{\mu'} . \tag{1.2}$$

The functions $g_i(\mu)$ with defines the RG flow of the theory in the space of couplings. The RG flow is conventionally thought of as being towards the IR, *i.e.* decreasing μ , but we shall often think about it in the other direction as well, towards the UV with μ increasing. In order that the RG equation (1.2) can hold it is necessary that the space of couplings includes *all* possible couplings (necessarily an infinite number). The RG is non-trivial because in order to lower the cut-off we somehow have to "integrate out" the degrees-of-freedom of the theory that lie between energy scales μ and μ' . In general this is a difficult step, however, as we shall see, in QFT we are in a very lucky situation due to the remarkable focusing properties of RG flows.

The RG energy scale μ plays a central rôle in the theory and it is important to understand what exactly it is. To start with we have identifed μ with the physical cut-off; however, there is another way to interpret μ . The point is that if we wish to describe physical process at the energy scales E_{phys} or below, or distance scales greater than $\ell = E_{\text{phys}}^{-1}$, then there is no reason why we cannot take the cut-off to be at the scale $\mu = E_{\text{phys}}$. In fact this would be the optimal choice since the effective description would then only involve modes with energies $\leq E_{\text{phys}}$, *i.e.* the ones directly involved in the physical process. So another to think of RG flow is that the couplings run with the typical energy scale of the process being investigated, $g_i(\mu)$, where μ is that energy scale.

Since the physical observables of the theory can be determined once the action is known, the RG transformation itself follows from following how the action changes as the cut-off changes. The action at a particular cut-off is known as the Wilsonian Effective Action $S[\phi; \mu, g_i]$ and since it depends on the fields, as well as the couplings, the RG transformation must be generalized to

The Key RG Equation $S[Z(\mu)^{1/2}\phi;\mu,g_i(\mu)] = S[Z(\mu')^{1/2}\phi;\mu',g_i(\mu')], \qquad (1.3)$

where $Z(\mu)$ is known as "wavefunction renormalization" of the field. In the general case with many fields, $Z(\mu)$ is a matrix quantity that can mix all the fields. The action of a QFT can be written as the sum of a kinetic term and linear combination of "operators" $\mathcal{O}_i(x)$ which are powers of the fields and their derivatives, e.g. ϕ^n , $\phi^n \partial_\mu \phi \partial^\mu \phi$, etc:²

$$S[\phi;\mu,g_i] = \int d^d x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sum_i \mu^{d-d_i} g_i \mathcal{O}_i(x) \right]$$
(1.4)

where d_i is the classical dimension of $\mathcal{O}_i(x)$. Notice that we chosen the couplings to be dimensionless by inserting the appropriate power of the cut-off to soak up the dimensions. This is because it is really the

value of a coupling relative to the cut-off that is physically relevant. The wavefunction renormalization factor $Z(\mu)$ can be thought of as the coupling to the kinetic term since³

$$S[Z^{1/2}\phi;\mu,g_i] = \int d^d x \left[\frac{Z}{2}\partial_\mu\phi\partial^\mu\phi + \cdots\right].$$
(1.5)

It is often useful to think about infinitesimal RG transformations, in which case we define the *beta* function of a theory

$$\mu \frac{dg_i(\mu)}{d\mu} . \tag{1.6}$$

The running couplings then follow by integration of the beta-function equations above. Notice that due to the fact that couplings always appear as combinations $\mu^{d-d_i}g_i$ means that the beta functions always have the form

$$\mu \frac{dg_i}{d\mu} = (d_i - d)g_i + \beta_{g_i}^{\text{quant.}} .$$
(1.7)

One can think of the first term as arising from classical scaling and the second piece as arising from the non-trivial integrating-out part of the RG transformation. If \hbar were re-introduced $\exp[iS] \to \exp[iS/\hbar]$, then the quantum piece would indeed vanish in the limit $\hbar \to 0$. We can also define the *anomalous dimension* of a field ϕ as

$$\gamma_{\phi} = -\frac{\mu}{2} \frac{d \log Z(\mu)}{d\mu} . \tag{1.8}$$

In particle physics the ultimate physical observables are the probabilities for particular things to happen. However, it is often useful, especially in massless theories where the S-matrix is problematic to formulate because of long-range interactions, to consider the Green functions of fields. Schematically,

$$\left\langle \phi(x_1) \cdots \phi_n(x_n) \right\rangle_{g_i(\mu),\mu} = \frac{\int_{\mu} [d\phi] \, e^{iS[\phi;\mu,g_i(\mu)]} \phi(x_1) \cdots \phi(x_n)}{\int_{\mu} [d\phi] \, e^{iS[\phi;\mu,g_i(\mu)]}} \,.$$
 (1.9)

It follows from (1.3) that for these quantities that depend on the field we must generalize (1.2) to take account of wavefunction renormalization, giving

$$Z(\mu)^{-n/2} \langle \phi(x_1) \cdots \phi_n(x_n) \rangle_{g_i(\mu),\mu} = Z(\mu')^{-n/2} \langle \phi(x_1) \cdots \phi_n(x_n) \rangle_{g_i(\mu'),\mu'} , \qquad (1.10)$$

What is particularly important about RG flows are their IR and UV limits; namely $\mu \to 0$ and $\mu \to \infty$, respectively. As we flow towards the IR, all masses relative to the cut-off, that is m/μ , increase. If a theory has a mass-gap (no massless particles) then as $\mu \to 0$ all physical masses are become infinitely heavy relative to the cut-off and there is nothing left to propagate in the IR. Hence, in the IR limit we have an empty, trivial or null theory. The other possibility is when the RG flow starts on the:

Critical Surface

The infinite dimensional subspace in the space-of-theories for which the mass gap vanishes. These theories consequently have a non-trivial IR limit in which only the massless degrees-of-freedom remain.

In this case, as $\mu \to 0$ the massless particles will remain and in all known cases the couplings flow to a fixed point of the RG $g_i(\mu) \to g_i^*$ as $\mu \to 0$ where the beta functions vanish:⁴

Equation for a fixed point or conformal field theory

$$\mu \frac{dg_i}{d\mu}\Big|_{g_j^*} = 0 . \tag{1.11}$$

The theories at the fixed points are very special because as well only having massless states particles they have no dimension-full parameters at all. This means that they are scale invariant. However, this scale invariance is naturally promoted to the group of conformal transformations and so the fixed point theories are also "conformal field theories" (CFTs).⁵

In the neighbourhood of a fixed point, or CFT, $g_i = g_i^* + \delta g_i$, we can alway linearize the RG flows:

$$\left. \mu \frac{dg_i}{d\mu} \right|_{g_j^* + \delta g_j} = A_{ij} \delta g_j + \mathcal{O}(\delta g_j^2) .$$
(1.12)

In a suitable diagonal basis for $\{\delta g_i\}$ which we denote $\{\sigma_i\}$,

$$\mu \frac{d\sigma_i}{d\mu} = (\Delta_i - d)\sigma_i + \mathcal{O}(\sigma^2)$$
(1.13)

and so to linear order the RG flow is simply

$$\sigma_i(\mu) = \left(\frac{\mu}{\mu'}\right)^{\Delta_i - d} \sigma_i(\mu') . \qquad (1.14)$$

The quantity Δ_i is called the scaling (or conformal) dimension of the operator associated to σ_i . In general in an interacting QFT it will not be the classical scaling dimension and the difference

$$\gamma_i = \Delta_i - d_i \tag{1.15}$$

is known as the anomalous dimension of the operator.

In a CFT the Green functions are covariant under scale transformations and this provides non-trivial constraints. As an example, consider the 2-point Green function $\langle \phi(x)\phi(0)\rangle$. This satisfies the more general RG equation (1.10)

$$Z(\mu)^{-1} \langle \phi(x)\phi(0) \rangle_{g_i(\mu),\mu} = Z(\mu')^{-1} \langle \phi(x)\phi(0) \rangle_{g_i(\mu'),\mu'} .$$
(1.16)

At a fixed point $g_i(\mu) = g_i(\mu') = g_i^*$ and we have $Z(\mu) = (\mu'/\mu)^{2\gamma_{\phi}^*}Z(\mu')$, where $\gamma_{\phi}^* = \gamma_{\phi}(g_i^*)$. Using dimensional analysis we must have

$$\left\langle \phi(x)\phi(0)\right\rangle_{g_i^*,\mu} = \mu^{2d_\phi}\mathcal{G}(x\mu) , \qquad (1.17)$$

where d_{ϕ} is the classical dimension of the field ϕ . Substituting into the RG equation allows us to solve for the unknown function \mathcal{G} , up to an overall multiplicative constant, yielding

$$\left\langle \phi(x)\phi(0)\right\rangle_{g_{i}^{*},\mu} = \frac{c}{\mu^{2\gamma_{\phi}^{*}}x^{2d_{\phi}+2\gamma_{\phi}^{*}}} \propto \frac{1}{x^{2\Delta_{\phi}^{*}}}$$
(1.18)

where c is a constant. This is the typical power-law behaviour characteristic of correlation functions in a CFT. In the problem for this lecture you will see that using the whole of the conformal group provides even more information.

Relevant, Irrelevant and Marginal

Couplings in the neighbourhood of a fixed point flow as (1.14) and are classified in the following way:

(i) If a coupling has $\Delta_i < d$ the flow diverges away from the fixed point into the IR as μ decreases and is therefore known as a *relevant* coupling.

(ii) If $\Delta_i > d$ the coupling flows into the fixed point and is known as *irrelevant*.

(iii) The case $\Delta_i = d$ is a marginal coupling for which one has to go to higher order to find out the behaviour. If, due to the higher order terms, a coupling diverges away/converges towards from the fixed point it is *marginally relevant/irrelevant*. The final possibility is that the coupling does not run to all orders. In this case it is *truly marginal* coupling and implies that the original fixed point is actually part of a whole line of fixed points.

When we follow an RG flow backwards towards the UV all particle masses decrease relative to cut-off and the so trajectory of a theory with a mass gap must approach the critical surface. In typical cases, with or without a mass-gap, the trajectory either diverges off to infinity for finite μ or approaches a fixed point lying on the critical surface in the limit $\mu \to \infty$. Below we show the RG flows around a fixed point with 2 irrelevant directions and 1 relevant direction.



Notice that the flows lying off the critical surface naturally focus onto the "renormalized trajectory", which is defined as the flow that comes out of the fixed point. The focusing effect, along with the fact that there are only a finite number of relevant directions, leads to the property of *Universality* which is the most important feature of RG flows. Universality also arises for flows starting on the critical surface since in this case they all flow into the fixed point.

Universality

CFTs only have a finite (and usually small) number of relevant couplings. This means that the domain-of-attraction of a fixed point (the set of all points in theory-space) that flow into a fixed point is infinite dimensional (this dimension is the number of irrelevant couplings). This also means that RG flows of a theory lying off the critical surface strongly focus onto finite dimensional subspaces parameterized by the relevant couplings of a fixed point as in the figure above. The implication of this is that the behaviour of theories in the IR is determined by only a small number of relevant couplings and not by the infinite set of couplings g_i . This means that IR behaviour of a given theory with couplings g_i can lie in a small set of "universality classes" which are determined by the domain of attraction of the set of fixed points.

Notice that the RG is directly relevant to the problem of taking a continuum limit of a QFT:

The Continuum Limit

This is the process of taking the cut-off from its original value μ to ∞ whilst keeping the keeping the physics at any energy less than the original μ fixed. Whether such a limit exists is a highly non-trivial issue and central to our story. Notice that taking a continuum limit involves the inverse RG flow, that is $g_i(\mu)$ with μ increasing.

The RG Equation (1.2) shows how this can be achieved. We can send $\mu \to \infty$, as long as the UV limit of $g_i(\mu)$ is suitably well-defined which in practice means that $g_i(\infty)$ is a fixed-point of RG (this is what Weinberg calls "asymptotic safety). The resulting $g_i(\mu)$ is known as a "renormalized trajectory" since it defines a theory on all length scales. Clearly a renormalized trajectory has to have the infinite set of irrelevant couplings at the UV fixed-point vanishing. Searching for a renormalized trajectory would seem to involve searching for a needle in an infinite haystack. Fortunately, however, universality comes to our rescue:

Taking a Continuum Limit

We do not need to actually sit precisely on the renormalized trajectory in order to define a continuum theory. All that is required is a one-parameter set of theories defined with cut-off μ' and with couplings $g_i = \tilde{g}_i(\mu')$ (which need not necessarily be an RG flow, since this would mean sitting on the renormalized trajectory) for which $\tilde{g}_i(\infty)$ lies in the domain of attraction of the UV CFT, as illustrated below, where the domain of attraction is the shaded area



The limit $\mu' \to \infty$ is defined in such a way that the IR physics at the original cut-off scale μ is fixed. In particular, the number of parameters that must be specified in order to take a continuum limit, *i.e.* which fix the IR physics, equals the number of relevant couplings of the CFT. However, both the way that relevant couplings are fixed at the scale μ and the values of the irrelevant couplings as $\tilde{g}_i(\mu')$ as $\mu' \to \infty$ can be defined in many different ways. So there are many ways to take a continuum limit, or many "schemes", which all lead to the same continuum theory. In particular, in particle physics this always allows us to take very simple forms for the action with a small number of operators (equal to the number of relevant coupling of the UV CFT). However, at the same time it allows us to describe the same QFT by using, say, a lattice cut-off.

Notes

1 In the literature, RG equations like (1.2) are also often written in infinitesimal form as

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_{g_i} \frac{\partial}{\partial g_i}\right] \mathscr{F}(g_i; \ell)_{\mu} = 0 . \qquad (1.19)$$

where $\beta_{g_i} = \mu dg_i / d\mu$.

2 The use of the term "operator" or "composite operator" derives from a canonical quantization approach to QFT in which one builds a Hilbert space and on which the fields becomes operators. It is conventional to use this language when using the functional integral approach when strictly-speaking the quantities are not operators.

3 You may have noticed that wavefunction renormalization of the fields in QFT plays a special role relative to the other coupling in the action. In a sense the definition of RG is ambiguous since the beta-functions of the couplings depend implicitly on $Z(\mu)$? Another more concrete way to phrase this question is why did we *choose* the wavefunction renormalization in order to keep the kinetic term intact after RG flow? The intuitive answer is that we want our theories to describe particles which, at least in the case of a theory with a mass gap, when well separated are approximately free and so have a free field propagator $\sim i/(p^2 - m_{\rm ph}^2)$. This is why we re-scaled the field in the effective action to keep the kinetic term like that for a free particle. The physical mass of the particle $m_{\rm ph}$ (which is not the parameter m in the original action) is then identified by the position of the pole in the propagator.

In the statistic physics interpretation of the functional integral, other choices of wavefunction renormalization are sometimes appropriate. For instance, in $d \ge 4$ one can impose a choice where the kinetic term is relevant leaving a theory with no propagating field. In this case, non-zero momentum modes of the field are suppressed in the IR and only the zero mode, which is constant in space, survives. This limit is known as *mean field theory* in statistical physics.

4 Other more exotic possibilities, like limit cycles, have been found in some very bizarre theories in two spacetime dimensions.

5 One potential point of confusion regarding fixed points of the RG flow is that the fields can still have non-trivial wavefunction renormalization factors $Z(\phi)$. However, remember that the fields are in some sense just dummy variables. So a fixed point theory, or CFT, is really scale *covariant* rather than scale *invariant*.

The (connected part of the) conformal group consists of Poincaré transformations along with scale transformations, or "dilatations" $x \to sx$, and special conformal transformations

$$x \longrightarrow \frac{x + x^2 b}{1 + 2b \cdot x + x^2 b^2} \tag{1.20}$$

The infinitesimal transformations for Lorentz, dilatations and special conformal transformations are

$$\delta x^{\mu} = \epsilon^{\mu}{}_{\nu}x^{\nu}$$
, $\delta x^{\mu} = sx^{\mu}$, $\delta x^{\mu} = x^{2}b^{\mu} - 2x^{\mu}(x \cdot b)$, (1.21)

where $\epsilon^{\mu\nu} = -\epsilon^{\mu\nu}$. In any local QFT there exists an energy-momentum tensor $T_{\mu\nu}$ and correlation functions satisfy the Ward identity

$$\sum_{p=1}^{n} \left\langle \phi_1(x_1) \cdots \delta \phi_p(x_p) \cdots \phi_n(x_n) \right\rangle = -\int d^d x \left\langle \phi_1(x_1) \cdots \phi_n(x_n) T^{\mu}{}_{\nu}(x) \right\rangle \partial_{\mu} \left(\delta x^{\nu} \right) \,. \tag{1.22}$$

Invariance of the QFT under Lorentz transformations requires $T_{\mu\nu} = T_{\nu\mu}$ while invariance under dilatations $T^{\mu}{}_{\mu} = 0$. From this it follows that these two conditions are sufficient to imply invariance under infinitesimal special conformal transformations.

2 Scalar Field Theories

We now illustrate RG theory in the context of the QFT of a single scalar field. Usually we write down simple actions like

$$S[\phi] = \int d^d x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2 \phi^2 - \frac{1}{4!}\lambda \phi^4\right), \qquad (2.1)$$

however, in the spirit of RG we should allow all possible operators consistent with spacetime symmetries. In the case of a scalar field, all powers of the field and its derivatives, where the latter are contracted in a Lorentz invariant way. For simplicity we shall restrict to operators even in $\phi \to -\phi$.¹. Simple scaling analysis shows that a "composite operator" \mathcal{O} containing p derivatives and 2n powers of the field, schematically $\partial^p \phi^{2n}$, has classical dimension

$$d_{\mathcal{O}} = n(d-2) + p \;. \tag{2.2}$$

Even at the classical level we see that the number of relevant/marginal couplings, those with $d_{\mathcal{O}} \leq d$ is small. The classical dimensions of various operators are given in the table below.

\mathcal{O}	d > 4	d = 4	d = 3	d = 2
ϕ^2	rel	rel	rel	rel
ϕ^4	irrel	marg	rel	rel
ϕ^6	irrel	irrel	marg	rel
ϕ^{2n}	irrel	irrel	irrel	rel
$\partial \phi_{\mu} \partial^{\mu} \phi$	marg	marg	marg	marg

The classical scaling suggests that we only need keep track of the kinetic term and potential,²

$$\mathscr{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi) , \qquad (2.3)$$

where we take

$$V(\phi) = \sum_{n} \mu^{d-n(d-2)} \frac{g_{2n}}{(2n)!} \phi^{2n} , \qquad (2.4)$$

and where we have used the cut-off to define dimensionless couplings g_{2n} .

Now we come to crux of the problem, that of finding the RG flow, or concretely the beta functions, of the couplings. In order to do this we must apply the RG equation (1.3) to the Wilsonian Effective Action $S[Z(\mu)^{1/2}\varphi; \mu, g_i(\mu)]$ defined for the theory with cut-off μ in such a way that the phenomena on energy scales below the cut-off is fixed as μ is varied.

Before we can describe how to relate the theories with cut-off μ and μ' let us first choose a cut-off procedure. The most basic and conceptually simple way is to introduce a sharp momentum cut-off on

the Fourier modes after Wick rotation to Euclidean space. In Euclidean space the Lagrangian (2.3) has the form

$$\mathscr{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 + V(\phi) \tag{2.5}$$

and the functional integral becomes $\int [d\phi] e^{-S}$.³ The momentum cut-off involves Fourier transforming the field

$$\phi(x) = \int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) e^{ip \cdot x}$$
(2.6)

and then limiting the momentum vector by a sharp cut-off $|p| \leq \mu$. The resulting theory is manifestly UV finite since loop integrals can never diverge. In addition, we have a very concrete way of performing the RG transformation. Namely, we split the field ϕ defined with cut-off μ' into

$$\phi = \varphi + \hat{\phi} , \qquad (2.7)$$

where φ has modes with $|p| \leq \mu$ while $\hat{\phi}$ are modes with $\mu \leq |p| \leq \mu'$. In order to extract the betafunction it is sufficient to consider the infinitesimal transformation with $\mu' = \mu + \delta \mu$. We can then obtain the RG flow by considering how the action changes when we integrating out $\hat{\phi}$, so concretely

$$\exp\left\{-S_{\text{eff}}[\varphi]\right\} = \int [d\hat{\phi}] \exp\left\{-S[\varphi + \hat{\phi}; \mu', g_{2n}(\mu')]\right\}.$$
(2.8)

On the left-hand side we have the Wilsonian Effective Action which is to be identified with

$$S_{\text{eff}}[\varphi] = S[Z(\mu)^{1/2}\varphi; \mu, g_{2n}(\mu)] .$$
(2.9)

Notice that we have taken $Z(\mu') = 1$ since we only need the variation as μ changes in order to extract the anomalous dimension γ_{ϕ} .

Expanding the action on the right-hand side in powers of $\hat{\phi}$:⁴

$$S[\varphi + \hat{\phi}] = S[\varphi] + \int d^d x \left(\frac{1}{2} (\partial_\mu \hat{\phi})^2 + \frac{1}{2} V''(\varphi) \hat{\phi}^2 + \frac{1}{6} V'''(\varphi) \hat{\phi}^3 + \cdots \right) .$$
(2.10)

Feynman Diagram Interpretation

Contributions to the effective action can be interpreted in term of Feynman diagrams with only $\hat{\phi}$ on internal lines with a propagator $1/(p^2 + g_2\mu^2)$, with p integrated over the shell $\mu \leq |p| \leq \mu'$, and with only φ on external lines (but amputated meaning no propagators on the external lines). The vertices are provided by the interaction terms in $V(\varphi + \hat{\phi})$. Each loop involves an integral over the momentum of $\hat{\phi}$ which lies in a shell between radii μ and μ' in momentum space:

$$\int_{\mu \le |p| \le \mu'} \frac{d^d p}{(2\pi)^d} f(p) \ . \tag{2.11}$$

However, if we are only interested in an infinitesimal RG transformation $\mu' = \mu + \delta \mu$ then the integrals over internal momenta (2.11) become much simpler:

$$\int_{\mu \le |p| \le \mu + \delta\mu} \frac{d^d p}{(2\pi)^d} f(p) = \frac{\mu^{d-1}}{(2\pi)^d} \int d^{d-1} \hat{\Omega} f(\mu \hat{\Omega}) \,\delta\mu \,\,, \tag{2.12}$$

where $\hat{\Omega}$ is a unit *d* vector integrated over a unit S^{d-1} . In addition, since each loop integral brings a factor of $\delta\mu$, so to linear order in $\delta\mu$ only one loop diagrams are needed. This is equivalent to saying that we only need the term quadratic in $\hat{\phi}$ in (2.10). The resulting integral over $\hat{\phi}$ is Gaussian and yields⁵

$$S_{\text{eff}}[\varphi] = S[\varphi] + \frac{1}{2}\log\det\left(-\Box + V''(\varphi)\right).$$
(2.13)

In order to extract the RG transformation we identify the left-hand side with the Wilsonian effective action (2.9). In this case, one finds that no wavefunction renormalization is required.⁶ In order to extract the effective potential (that is the part of the Lagrangian not involving derivatives of the field) we can temporarily assume that φ is constant, in which case

$$\frac{1}{2}\log\det\left(-\Box+V''(\varphi)\right) = a\mu^{d-1}\int d^dx\,\log(\mu^2+V''(\varphi))\,\delta\mu\;,\tag{2.14}$$

where $a = \text{Vol}(S^{d-1})/(2(2\pi)^d) = 2^{-d}\pi^{-d/2}/\Gamma(d/2)$. From this expanding in powers of φ it follows that

$$\mu \frac{dg_{2n}}{d\mu} = \left(n(d-2) - d \right) g_{2n} - a\mu^{2n} \frac{d^{2n}}{d\varphi^{2n}} \log(\mu^2 + V''(\varphi)) \Big|_{\varphi=0} .$$
(2.15)

The contributions on the right-hand side are identified with one-loop diagrams of the form



with 2n external legs.

From (2.15), it follows, for example, that

$$\mu \frac{dg_2}{d\mu} = -2g_2 - \frac{ag_4}{1+g_2} ,$$

$$\mu \frac{dg_4}{d\mu} = (d-4)g_4 + \frac{3ag_4^2}{(1+g_2)^2} - \frac{ag_6}{1+g_2} ,$$

$$\mu \frac{dg_6}{d\mu} = (2d-6)g_6 - \frac{30ag_4^3}{(1+g_2)^3} + \frac{15ag_4g_6}{(1+g_2)^2} - \frac{ag_8}{1+g_2} .$$
(2.16)

Notice that the quantum contributions involve inverse powers of the factor $1 + g_2$ which physically is $m^2/\mu^2 + 1$. So when $m \gg \mu$, the quantum terms are suppressed as one would expect on the basis of decoupling.

Decoupling

Decoupling expresses the intuition that a particle of mass m cannot directly affect the physics on energy scales $\ll m$. For instance, the potential due to the exchange of massive particle in 4 dimensions is $\sim e^{-mr}/r$. This is exponentially suppressed on distances scales $\gg m^{-1}$.

The beta functions allow us to map-out RG flow on theory space. The first thing to do is to find the RG fixed points corresponding to the CFTs. The "Gaussian" fixed point is the trivial fixed point where all the couplings vanish. Linearizing around this point, the beta-functions are

$$\mu \frac{dg_{2n}}{d\mu} = \left(n(d-2) - d \right) g_{2n} - ag_{2n+2} .$$
(2.17)

So the scaling dimensions are the classical dimensions $\Delta_{2n} = d_{2n} = n(d-2)$, *i.e.* the anomalous dimensions vanish, although the couplings that diagonalize the matrix of scaling dimensions σ_{2n} are not precisely equal to g_{2n} due to the second term in (2.17). In particular, $\sigma_2 = g_2$ is always relevant, $\sigma_4 = g_4 + ag_2/(2-d)$ is relevant for d < 4, irrelevant for d > 4 and marginally irrelevant for d = 4. In this latter case we need to go beyond the linear approximation. Since g_6 is irrelevant in d = 4, we shall ignore it, in which case since $a = 1/(16\pi^2)$ we have

$$\mu \frac{dg_4}{d\mu} = \mu \frac{dg_4}{d\mu} = \frac{3}{16\pi^2} g_4^2 , \qquad (2.18)$$

whose solution is

$$\frac{1}{g_4(\mu)} = C - \frac{3}{16\pi^2} \log \mu .$$
(2.19)

This shows that g_4 is actually marginally irrelevant at the Gaussian fixed point because it gets smaller as μ decreases. We usually write the integration constant in terms of a dimensionful parameter Λ as

$$g_4(\mu) = \frac{16\pi^2}{3\log(\Lambda/\mu)} , \qquad (2.20)$$

with $\mu < \Lambda$. This is our first example of *dimensional transmutation* where the degree-of-freedom of a dimensionless coupling g_4 has changed into a dimensionful quantity, namely Λ .

To find other non-trivial fixed points is difficult and the only way we can make progress is to work perturbatively in the couplings. This turns out to be consistent only if we accept the perversion of working in arbitrary non-integer dimension and regard $\epsilon = 4 - d$ as a small parameter. In that case, we find a new non-trivial fixed point known as the Wilson-Fischer fixed point at

$$g_2^* = -\frac{1}{6}\epsilon + \cdots, \qquad g_4^* = \frac{1}{3a}\epsilon + \cdots, \qquad g_{2n>4}^* \sim \epsilon^n + \cdots$$
 (2.21)

In particular, the Wilson-Fischer fixed point is only physically acceptable if $\epsilon > 0$, or d < 4, since otherwise the couplings g_{2n}^* are all negative and the potential of the theory would not be bounded from below. In the neighbourhood of the fixed point in the g_2, g_4 subspace we have to linear order in ϵ

$$\mu \frac{d}{d\mu} \begin{pmatrix} \delta g_2 \\ \delta g_4 \end{pmatrix} = \begin{pmatrix} \epsilon/3 - 2 & -a(1 + \epsilon/6) \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \delta g_2 \\ \delta g_4 \end{pmatrix} , \qquad (2.22)$$

with

$$a = 1/(16\pi^2) + \epsilon(1 - \gamma_E + \log 4\pi)/32\pi^2 + \mathcal{O}(\epsilon^2) . \qquad (2.23)$$

So the scaling dimensions of the associated operators and the associated couplings are

$$\Delta_2 = 2 - 2\epsilon/3 , \qquad \sigma_2 = \delta g_2 ,$$

$$\Delta_4 = 4 , \qquad \sigma_4 = \delta g_4 - \frac{a}{2 + \epsilon/3} \delta g_2 .$$
(2.24)

So at this fixed point only the mass coupling is relevant.

The flows in the (g_2, g_4) subspace of scalar QFT for small $\epsilon > 0$ are shown below



The Gaussian and Wilson-Fischer fixed points are shown and all the other couplings are irrelevant and so flow to the (g_2, g_4) subspace. Notice that the critical surface intersects this subspace in the line that joins the two fixed points as shown

Although we have only proved the existence of the Wilson-Fischer fixed point for small ϵ , it is thought to exist in both d = 3 and d = 2. In the language of statistical physics it lies in the universality class of the Ising Model.⁷ What our simple analysis fails to show is that in d = 2 there are actually an infinite sequence of additional fixed points.⁸

Vacuum Expectation Values

In a scalar QFT, the field can develop a non-trivial Vacuum Expectation Value (VEV) $\langle \phi \rangle \neq 0$. This possibility is determined by finding the minima of the *effective* potential. This is the potential on the constant (or zero) mode of the field after all the non-zero modes have been integrated out. In other words, this is the potential in the Wilsonian effective action in the limit $\mu \to 0.9$ A VEV develops when the effective potential develops minima away from the origin as in



Notice that since we started with a theory symmetric under $\phi \to -\phi$, there will necessarily two possible vacuum states with opposite values of $\langle \phi \rangle$. A QFT must choose one or the other and so we say that the symmetry $\phi \to -\phi$ is spontaneously broken.¹⁰

Now that we have a qualitative picture of the RG flows, it is possible to describe the possible continuum limits of scalar field theories:

 $d \ge 4$: In this case, only the Gaussian fixed point exists and this fixed point only has one relevant direction, namely the mass coupling g_2 . Hence there is a single renormalized trajectory on which $g_2(\mu) = (\mu'/\mu)^2 g_2(\mu')$ while all the other couplings vanish. This renormalized trajectory describes the free massive scalar field. If we sit precisely at the fixed point we have a free massless scalar field. In particular, according to this analysis there is no interacting continuum theory in d = 4.

d < 4: At least for small enough ϵ (whatever that means) there are two fixed points and a twodimensional space of renormalized trajectories parameterized by the couplings g_2 and g_4 on which g_{2n} , n > 2 have some values fixed by g_2 and g_4 . In particular, if we parameterize our continuum theories by the values of g_2 and g_4 then they are limited to the region shown below



In particular, theory (a) is free and massive (and must have $g_2 > 0$); (b,d) is interacting and massive and in the UV becomes free since the trajectories originate from the Gaussian fixed point. In case (d) $g_2 < 0$ and the field has a VEV; (c,e) describe a massive interacting theory that becomes the Ising model universality class in the UV, case (e) has $g_2 < 0$ and a VEV; (f) describes a massless interacting theory that interpolates between a free theory in the UV and the Ising Model in the IR; (g) is a free massless theory; and (h) is the Ising model CFT.

Notes

1 This is an example of using a symmetry to restrict theory space. The important point is that the symmetry is respected by RG flow and so is self-consistent.

2 Note, however, that, in principle, there is no difficulty in keeping track of all the couplings including the higher derivative terms.

3 We take it as established fact that one can move between the Minkowski and Euclidean versions of the theory without difficulty. In our conventions, when we Wick rotate $g_{\mu\nu}a^{\mu}b^{\nu} = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \rightarrow -a_{\mu}b_{\mu} = -a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3$. In Euclidean space the functional integral $\int [d\phi]e^{-S[\phi]}$ can be interpreted as a probably measure (when properly normalized) on the the field configuration space. This is why Euclidean QFT is intimately related to systems in statistical physics.

4 In the following we have ignored the term which is linear in $\hat{\phi}$. The reason is that we are after an effective action for φ which is local (that is a spacetime integral over terms which are powers of the field and its derivatives). The linear coupling in $\hat{\phi}$ gives rise to graphs which are "1-particle reducible", the simplest of which is shown below



The contribution from this graph is only non-vanishing if $\mu \leq |p_1 + p_2 + p_3| \leq \mu + \delta \mu$. In particular as long as we work with local actions which are expansions in derivatives, *i.e.* momenta, we can

ignore such contributions. These non-analytic contributions are the price we pay for working with a sharp momentum cut-off. It is important to understand that although the effective action would have these non-analytic contributions the observables of the theory, namely the Green functions or the S-matrix, would be perfectly well behaved. For a discussion of this in much greater detail, see the excellent lecture notes of Weinberg.

5 Here we used the identity

$$\int d^n x \, e^{-x \cdot A \cdot x} = \frac{\pi^{n/2}}{\det A} = \pi^{n/2} e^{-\frac{1}{2} \log \det A} \tag{2.25}$$

for a finite matrix A and extended it to the functional case.

6 The terms quadratic in ϕ come from the diagram below



which gives a contribution

$$\sim \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(k) \tilde{\phi}(-k) \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + g_2 \mu^2} = \text{const.} \times \int d^d x \, \phi(x)^2 \,. \tag{2.26}$$

Notice that the resulting expression does not involve derivatives of the field because, due to momentum conservation, no external momentum can flow in the loop. Hence, there is no wavefunction renormalization.

7 The Ising Model is a statistical model defined on a square lattice with spins $\sigma_i \in \{+1, -1\}$ at each site and with an energy (which we identify with the Euclidean action)

$$\mathscr{E} = -\frac{1}{T} \sum_{(i,j)} \sigma_i \sigma_j . \qquad (2.27)$$

The sum is over all nearest-neighbour pairs (i, j) and T is the temperature. Notice that at low temperatures, the action/energy favours alignment of all the spins, while at high temperatures thermal fluctuations are large and the long-range order is destroyed. This can be viewed as a competition between energy and entropy. There is a 2nd order phase transition at a critical temperature $T = T_c$ at which there are long-distance power-law correlations. This critical point is in the same universality class as the Wilson-Fischer fixed point and the water-steam critical point.

8 In d = 2 there are powerful methods for analyzing CFTs because in d = 2 the conformal group is infinite dimensional since it consists of any holomorphic transformation $t \pm x \rightarrow f(t \pm x)$.

9 It can be shown that the effective potential defined in terms of the Wilsonian effective action is equal to the effective potential extracted from the more familiar 1-Particle Irreducible (1-PI) effective action defined in perturbation theory—at least for a QFT with a mass gap. In the massless case the latter quantity is ill-defined due to IR divergences.

10 The reason why spontaneous symmetry breaking occurs is that the zero mode of a scalar field is not part of the variables that are integrated over in the measure $\int [d\phi]$, rather it acts as a boundary condition on the scalar field at spatial infinity. However, this is only true in spacetime dimensions d > 2: in d = 2 one must integrate over the zero mode and so spontaneous symmetry breaking of *continuous* symmetries cannot occur. Of course $\phi \to -\phi$ is a discrete symmetry that can still be spontaneously broken even in d = 2.

3 RG and Perturbation Theory

In this lecture, we consider how to implement RG ideas in the context of perturbation theory and in a way which is easier to generalize to other theories.

It is the principle of universality that allows us to formulate our theories in terms of simple actions. All we need do is include the relevant couplings: all the irrelevant couplings can be taken to vanish. So—at least for d > 2—it is sufficient to write the simple action in (2.1).¹ In order to take a continuum limit, we need to let the "bare couplings" $g_i(\mu')$ to depend on the cut-off μ' in such a way that the physics at a physically relevant scale $\mu < \mu'$ remains fixed. This is guaranteed if we use the RG equation (1.2) and take $\mu' \to \infty$ with the couplings $g_i(\mu')$ following the RG flow into the UV. However, this would mean keeping all the couplings to irrelevant operators. As we described in Lecture 1, there is no need to do this. It is sufficient to keep only the relevant couplings $g_2(\mu')$ and $g_4(\mu')$ (as well as the kinetic term) and choose all the irrelevant couplings to vanish. So in this sense, the continuum limit flow is not strictly speaking an RG flow; rather it is the RG flow restricted to the space of relevant couplings.

Perturbation Theory: The Facts of Life

Now we come to a very important issue. In real life we have to rely almost always on perturbation theory in the couplings. But according to the RG the couplings flow and this begs the question as to which coupling should use to perform perturbation theory? For example in d = 4, g_4 runs according to (2.19). So the coupling at the scale μ' is an infinite perturbative series of the coupling at μ :

$$g_4(\mu') = \frac{g_4(\mu)}{1 + \frac{3}{16\pi^2}g_4(\mu)\log\mu'/\mu} = g_4(\mu) - \frac{3}{16\pi^2}g_4(\mu)^2\log\mu'/\mu + \cdots$$
(3.1)

If we could completely sum all of the perturbative expansion in $g_4(\mu)$, then the resulting physical observables would—by definition—be independent of μ . However, since perturbation theory involves a truncation it is clear that perturbation theory in $g_4(\mu)$ will be different from perturbation theory in $g_4(\mu')$. Do we have a choice of μ and can we choose μ so that $g_4(\mu)$ is small and perturbation theory is reliable? The answer is practically no: the appropriate choice of μ is dictated by the characteristic physical scale involved and we just have to hope that at this scale $g_4(\mu)$ is small. If we try to do perturbation at a non-physical scale then any improvement we might make in having a smaller $g_4(\mu)$ is compensated by larger coefficients multiplying the powers of the coupling. We will see an example of how the physical scale dictates the choice of μ in a procedure known as *renormalization group improvement*.

So we should perform the perturbation expansion in the coupling at the physically relevant energy scale μ , *i.e.* in the "renormalized coupling" $\lambda \equiv \lambda(\mu)$ (we choose to use λ and m rather than g_4 and g_2 in the following description) rather than the "bare coupling" $\lambda_b \equiv \lambda(\mu')$ since $\lambda(\mu) < \lambda(\mu')$. To this

end, we split up the bare Lagrangian into the "renormalized" part and the "counter-term" part:

$$\mathscr{L} = \frac{1}{2} (\partial_{\mu} \phi_b)^2 + \frac{1}{2} m_b^2 \phi_b^2 + \frac{1}{4!} \lambda_b \phi_b^4 = \mathscr{L}_r + \mathscr{L}_{ct}$$
(3.2)

where \mathscr{L}_r has the same form as the bare Lagrangian, but with bare quantities replaced by renormalized ones (which we denote ϕ , m and λ):

$$\mathscr{L}_{r} = \frac{1}{2} (\partial_{\mu} \phi)^{2} + \frac{1}{2} m^{2} \phi^{2} + \frac{1}{4!} \lambda \phi^{4}$$
(3.3)

and \mathscr{L}_{ct} , the counter-term Lagrangian,

$$\mathscr{L}_{ct} = \frac{1}{2} \delta Z (\partial_{\mu} \phi)^2 + \frac{1}{2} \delta m^2 \phi^2 + \frac{1}{4!} \delta \lambda \phi^4 .$$
(3.4)

such that the bare quantities are

$$m_b^2 = m^2 + \delta m^2$$
, $\lambda_b = \lambda + \delta \lambda$, $\phi_b = \sqrt{1 + \delta Z} \phi$. (3.5)

It is important to emphasize that although we have denoted the counter-terms as $\delta \#$ they certainly are *not* infinitesimals.

Perturbation theory can be be performed in the renormalized coupling λ and the counter-terms found order-by-order in λ without prior knowledge of the RG flow. In this point-of-view, the counterterms are chosen to cancel the divergences that occur in the limit $\mu' \to \infty$.² For instance, in d < 4, the only Feynman diagrams which are superficially divergent³ have 2 external legs and are shown below:



While in d = 4 (and also for small $\epsilon = 4 - d$) all diagrams with 2 and 4 external lines are superficially divergent.

While we could continue with the sharp momentum cut-off, it is time to acknowledge its frailties. It has been a useful device to introduce the concept of RG flow, however, when we start to investigate gauge theories we discover that it is not obvious how to make the naïve sharp momentum cut-off consistent with gauge invariance. Fortunately, our all-dimension treatment of the scalar field naturally leads to the most important regularization scheme—at least within perturbation theory—known as *dimensional regularization*. The idea is to use the fact that we have defined the theory in $d = 4 - \epsilon$ dimensions and treat ϵ as a variable in its own right. Then, after performing the loop integrals, we will have analytic

functions of ϵ . The divergences in the physical dimensions $d = 2, 3, 4 \dots$ show up as poles as $\epsilon \to 2, 1, 0$. For example, in four dimensions we make the replacement

$$\int \frac{d^4 p}{(2\pi)^4} \longrightarrow \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} , \qquad (3.6)$$

where μ is a parameter, with unit mass dimension, which is introduced to make sure that the momentum integrals have the correct dimension. We will see that in a subtraction scheme known as *minimal subtraction* it plays a rôle analogous to the Wilsonian cut-off scale μ which is the scale at which the renormalized coupling are defined.

As an example consider the quadratically divergent one-loop graph in d = 4



which involves the loop integral

$$\lambda \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \cdot \frac{1}{p^2 + m^2} = \lambda \mu^{4-d} \frac{\operatorname{Vol}S^{d-1}}{(2\pi)^d} \int_0^\infty \frac{p^{d-1}dp}{p^2 + m^2}$$

$$= \lambda 2^{-d} \pi^{-d/2} \mu^{4-d} m^{d-2} \Gamma(1 - d/2) .$$
(3.7)

In the limit $\epsilon = 4 - d \rightarrow 0$ this equals

$$-\frac{\lambda m^2}{8\pi^2\epsilon} + \frac{\lambda m^2}{16\pi^2} \Big[-1 + \gamma_E - \log \frac{4\pi\mu^2}{m^2} \Big] + \mathcal{O}(\epsilon) .$$
(3.8)

The divergences can be removed by a mass counter-term of the form

$$\delta m^2 = \frac{\lambda m^2}{16\pi^2 \epsilon} + \frac{\lambda m^2}{32\pi^2} \left(-\gamma_E + \log 4\pi \right) \,. \tag{3.9}$$

As we have previously stated there is considerable freedom in removing the divergences. The choice (3.9) is known as *modified minimal subtraction*, denoted $\overline{\text{MS}}$. Here "minimal" refers to the singular part in (3.9) and "modified" refers to the second and third terms which are optional extras but part of the perturbative industry standard.

The other divergent diagram at one-loop is



In this case the loop integral is logarithmically divergent in d = 4:

$$\lambda^2 \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \cdot \frac{1}{p^2 + m^2} \cdot \frac{1}{(p+k)^2 + m^2} , \qquad (3.10)$$

where $k = k_1 + k_2$. Expanding in terms of the external momentum k it is only the term k = 0 which is divergent since every power of k effectively gives one less power of p for large p. The divergent piece is then

$$\lambda^{2} \mu^{4-d} \int \frac{d^{d}p}{(2\pi)^{d}} \cdot \frac{1}{(p^{2}+m^{2})^{2}} = \lambda^{2} \mu^{4-d} \frac{\operatorname{Vol} S^{d-1}}{(2\pi)^{d}} \int_{0}^{\infty} \frac{p^{d-1} dp}{(p^{2}+m^{2})^{2}} = \lambda^{2} 2^{-d} \pi^{-d/2} \mu^{4-d} m^{d-4} \Gamma(2-d/2) = \frac{\lambda^{2}}{8\pi^{2}\epsilon} + \frac{\lambda^{2}}{16\pi^{2}} \Big[-\gamma_{E} + \log \frac{4\pi\mu^{2}}{m^{2}} \Big] + \mathcal{O}(\epsilon) .$$
(3.11)

In minimal subtraction the divergence can be cancelled by the counter-term

$$\delta\lambda = \frac{3\lambda^2}{16\pi^2\epsilon} + \frac{3\lambda^2}{32\pi^2} \left(-\gamma_E + \log 4\pi\right) \,. \tag{3.12}$$

Later we shall see that regularization schemes like minimal subtraction are very different in character from alternative and more physically motivated schemes.⁴

As mentioned above, the renormalized couplings will depend on the energy scale μ in a way that is analogous to μ of the sharp momentum cut-off scheme. In fact in the we can recover the spirit of Wilson's RG by using what is known as the *background field method*. The idea is to expand the field ϕ in the renormalized action in terms of a slowly varying background field and a more rapidly fluctuating part:

$$\phi = \varphi + \hat{\phi} \ . \tag{3.13}$$

This is analogous to the decomposition (2.7) in the sharp momentum cut-off scheme. We then treat $\hat{\phi}$ as the field to integrate over in the functional integral whilst treating φ as a fixed background field.

Since φ is slowly varying we can expand in powers of derivatives of φ and as in Lecture 2, and in order to calculate the effective potential we can take φ to be constant. The effective potential in d = 4 to one loop is then

$$V_{\text{eff}}(\varphi) = V(\varphi) + \frac{1}{2}\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \log\left(p^2 + V''(\varphi)\right) = V(\varphi) - \frac{\left(V''(\varphi)\right)^2}{32\pi^2\epsilon} - \frac{\left(V''(\varphi)\right)^2}{64\pi^2} \left[\frac{3}{2} - \gamma_E + \log\frac{4\pi\mu^2}{V''(\varphi)}\right] .$$
(3.14)

Subtracting the divergence with a counter-term in the $\overline{\text{MS}}$ scheme gives

$$V_{\rm eff}(\varphi) = V(\varphi) - \frac{\left(V''(\varphi)\right)^2}{64\pi^2} \left[\frac{3}{2} + \log\frac{\mu^2}{V''(\varphi)}\right].$$
(3.15)

With a potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$, notice that the counter-term is proportional to

$$\left(V''(\varphi)\right)^2 = \left(m^2 + \frac{1}{2}\lambda\varphi^2\right)^2 \tag{3.16}$$

which has the same form as the original potential. In other words, we only need to add counter-terms for m^2 and λ . With a little more work one finds that the counter-terms are precisely (3.9) and (3.12).

The RG flow of the dimensionless couplings $g_2 = m^2/\mu^2$ and $g_4 = \lambda$ can be deduced, as before, by requiring that the effective potential satisfies an RG equation. For instance, in the form (1.3),

$$V_{\text{eff}}(Z(\mu)^{1/2}\varphi;\mu,g_i(\mu)) = V_{\text{eff}}(Z(\mu')^{1/2}\varphi;\mu',g_i(\mu')) , \qquad (3.17)$$

where now μ is no longer the Wilsonian cut-off but is the mass scale introduced in dimensional regularization to "fix up" the dimensions.

To one-loop order there is no wavefunction renormalization and using the above, we find

$$\mu \frac{dg_2}{d\mu} = -2g_2 + ag_2g_4 ,$$

$$\mu \frac{dg_4}{d\mu} = 3ag_4^2 .$$
(3.18)

where $a = 1/16\pi^2$. Compare (3.18) with the momentum cut-off scheme (2.16) (with d = 4 and with the irrelevant couplings set to 0). The first difference is that with the latter scheme the betafunction is exact at the one-loop level, whereas (3.18) receives contributions to all loop orders. Secondly, the momentum cut-off scheme displays manifest decoupling when $m \gg \mu$, whereas, on the contrary, dimensional regularization with $\overline{\text{MS}}$ subtraction does not. In the next lecture we will explore this in more detail and explain how decoupling must be implemented by hand in the $\overline{\text{MS}}$ scheme. What these two schemes illustrate is that the actual RG flows depend in detail on the chosen scheme. However, it is a central feature of the theory that the "topological" properties of the flows, meaning existence of fixed points, or whether a coupling is relevant or irrelevant, is scheme-independent. For example, if we work in dimensions d < 4 rather than d = 4, then one can demonstrate the existence of the Wilson-Fischer fixed point.

Our analysis of scalar field theories leads to the following important conclusion for spacetime dimension d = 4. Since we have only proved the existence of the Gaussian fixed point, the only continuum theory is a free massive or massless theory. This lack of an interacting continuum scalar theory in 4 dimensions is known as *triviality*. But, you will say, haven't we been able to find a renormalizable theory in 4 dimensions order-by-order in the perturbative expansion: isn't this an interacting theory? However, something does go terrible wrong with perturbation theory and this is caused by the wish to keep the renormalized coupling finite whilst sending the cut-off $\mu' \to \infty$. We can see what goes wrong by looking at the flow of the coupling (3.1): we see that as μ' increases there is singularity at

$$\mu' = \mu e^{16\pi^2/3g_4(\mu)} , \qquad (3.19)$$

where the bare coupling diverges. This is known as a Landau pole (this terminology is properly associated with QED which, as we shall see, suffers the same fate). Of course this is exactly what we expected: the coupling $g_4(\mu)$ is irrelevant at the Gaussian fixed point and so RG flow into the UV diverges away from the fixed point. Now we see within the one-loop approximation that the flow actually diverges to infinity at a finite value. So the conclusion is, in the absence of a non-trivial fixed point, scalar QFT is not really a renormalizable theory in d = 4. This conclusion applies more or less to the scalar Higgs sector of the standard model, and so in this sense the standard model is not truly a renormalizable theory and so, in a sense, predicts its own demise.

Summary

We now sketch how to use perturbation theory (when valid) and the background field method to calculate the RG flows of the revelant couplings.

(1) Write down a Lagrangian $\mathscr{L}_r(\phi)$ with all the "relevant" couplings g_i^r .

(2) Split $\phi = \varphi + \hat{\phi}$ and calculate loop diagrams with external φ and internal $\hat{\phi}$ (more precisely only include diagrams which are *one particle irreducible* 1PI) and add counter-terms along the way to cancel divergences. It is important that the counter-terms have the same form as $\mathscr{L}_r(\varphi)$ (if not then you didn't identify the complete set of relevant couplings).

(3) The analogue of the Wilsonian effective action in the special subspace of theories parameterized by the relevant coupling is then

$$S[\varphi;\mu,g_i^r] = \int d^d x \left(\mathscr{L}_r(\varphi) + \mathscr{L}_{ct}(\varphi) + \mathscr{L}_{1\mathrm{PI}}(\varphi) \right) \,. \tag{3.20}$$

Note that this action will include all the irrelevant couplings as well, but these will not be independent rather they will depend on the relevant couplings.

(4) Extract the RG flows of the revelant couplings in $\mathscr{L}_r(\varphi)$ by imposing the RG equation (1.3).

Notes

1 In d = 2 the situation is more complicated and we have to allow for an arbitrary potential since all operators ϕ^n are potentially relevant.

2 There is considerable freedom in choosing the way that the divergent terms are cancelled. These choices should be thought of as part of the regularization *scheme*.

3 The question of whether a given diagram is divergent can be partially addressed by calculating the superficial degree of divergence \mathcal{D} . This is the power of the overall momentum dependence of the diagram: each propagator contributes $\sim p^{-2}$ and each loop integral $\sim p^d$. One can show for a ϕ^4 theory that for a diagram with L loops and E external lines

$$\mathcal{D} = (d-4)L + 4 - E . \tag{3.21}$$

Notice that when $\mathcal{D} \geq 0$ the diagram is divergent, however, the converse doesn't imply convergence since a sub-diagram may be superficially divergent. When a theory only has superficial divergences in a finite number of diagrams it is called *super-renormalizable*. For small $\epsilon = 4 - d$ we expand first in ϵ and later take the continuum limit $\mu \to \infty$. In this case, the superficially divergent diagrams include all those 2 or 4 external legs.

4 $\overline{\text{MS}}$ subtraction is not the only way to remove the divergences in dimensional regularization. A more physically motivated scheme involves fixing the physical mass and physical coupling defined in terms of the propagator and Green functions at some characteristic Euclidean momentum scale $\tilde{\mu}$:

$$\langle \phi(p_1)\phi(p_2) \rangle = \frac{1}{p_1^2 + m^2} \delta^{(4)}(p_1 + p_2) ,$$

$$\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \rangle = \lambda \delta^{(4)}(p_1 + p_2 + p_3 + p_4) ,$$
 (3.22)

when $p_i^2 = \tilde{\mu}^2$. If we Wick rotate back to Minkowski space these conditions occur at space-like momenta $p_i^2 = -\tilde{\mu}^2$.

In this kind of momentum-dependent scheme, the quantity $\tilde{\mu}$ plays the rôle of the RG scale. The counter-terms δg_i depend explicitly on $\tilde{\mu}$ and the beta functions are equal to

$$\tilde{\mu} \frac{d\delta g_i}{d\tilde{\mu}} , \qquad (3.23)$$

where as usual the g_i are made dimensionless by using $\tilde{\mu}$. It is important to realize, as we shall see in the next lecture, that the beta-functions in this scheme are very different from the ones in the $\overline{\text{MS}}$ scheme.

4 Gauge Theories and Running Couplings

In this lecture, we turn our attention to the RG properties of gauge theories. We begin with the simplest gauge theory; namely, QED. In order that QED has interesting RG flow we require it to be interacting and to achieve this end we couple it to a charged scalar. The scalar field must necessarily be complex and for simplicity we take a (Minkowski) Lagrangian without any self-interactions for ϕ :

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_{\mu}\phi|^2 - m^2|\phi|^2 , \qquad (4.1)$$

where $D_{\mu}\phi = \partial_{\mu}\phi + ieA_{\mu}\phi$ and $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The theory is invariant under gauge transformations

$$\phi(x) \to e^{ie\alpha(x)}\phi(x) , \qquad A_{\mu}(x) \to A_{\mu}(x) - \partial_{\mu}\alpha .$$
 (4.2)

We want to focus on the RG flow of the charge e. In order to do this it is actually more convenient to re-scale $A_{\mu} \rightarrow A_{\mu}/e$ so that the coupling now appears in front of the photon's kinetic term

$$\mathscr{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + |(\partial_{\mu} + iA_{\mu})\phi|^2 - m^2 |\phi|^2 .$$
(4.3)

The reason why this is a convenient choice in the context of RG is that under the flow the structure of the covariant derivative must remain intact otherwise gauge invariance would not be respected. This means that A_{μ} with the Lagrangian as above must undergo no wavefunction renormalization, rather we interpret any renormalization of the gauge kinetic term as a renormalization of the electric charge e^{1} .

In order to find the RG flow of the coupling we use the background field method and consider A_{μ} as a background field and treat ϕ as the fluctuating field that is to be integrating out in the functional integral. The interaction terms are

$$\mathscr{L}_{\text{int}} = iA_{\mu} \left(\phi \partial^{\mu} \phi^* - \phi^* \partial^{\mu} \phi \right) + A_{\mu} A^{\mu} |\phi|^2 .$$
(4.4)

At the one-loop level there are 2 relevant graphs shown below



The first one cannot depend on the external momentum k and its rôle is to cancel the k-independent contribution of the first. In dimensional regularization, and temporarily Wick rotating to Euclidean space, the second diagram yields

$$\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \cdot \frac{(2p+k)_{\mu}(2p+k)_{\nu}}{\left(p^2+m^2\right)\left((p+k)^2+m^2\right)} \,. \tag{4.5}$$

The standard trick for dealing with a product of propagators is to introduce Feynman parameters, which in this case involves the identity

$$\frac{1}{p^2 + m^2} \cdot \frac{1}{(p+k)^2 + m^2} = \int_0^1 dx \, \frac{1}{\left((p+xk)^2 + x(1-x)k^2 + m^2\right)^2} \tag{4.6}$$

and then shifting $p \to p - xk$, in which case (4.5) becomes

$$\mu^{4-d} \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \cdot \frac{(2p + (1-2x)k)_\mu (2p + (1-2x)k)_\nu}{(p^2 + x(1-x)k^2 + m^2)^2} = \mu^{4-d} \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} \int_0^1 dx \,\Delta^{d/2-2} \Big[(1-d/2)(1-2x)^2 k_\mu k_\nu + 2\Delta\delta_{\mu\nu} \Big]$$
(4.7)

where $\Delta = x(1-x)k^2 + m^2$. The second term in the bracket can be re-written using the identity

$$2\int_{0}^{1} dx \,\Delta^{d/2-1} = k^2 (d/2 - 1) \int_{0}^{1} dx \,(1 - 2x)^2 \Delta^{d/2-2} + 2m^{d-2} \,. \tag{4.8}$$

The last term here cancels the contribution

$$\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{\delta_{\mu\nu}}{p^2 + m^2} = \mu^{4-d} \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} m^{d-2}$$
(4.9)

from the first diagram above. This leaves

$$\frac{\mu^{4-d}}{(4\pi)^{d/2}} \left(k_{\mu}k_{\nu} - k^{2}\delta_{\mu\nu}\right) \Gamma(2 - d/2) \int_{0}^{1} dx \left(1 - 2x\right)^{2} \left(x(1 - x)k^{2} + m^{2}\right)^{d/2 - 2} .$$
(4.10)

The structure of this result is very significant because if we expand in powers of k then a given term

$$k^{2n} \left(k_{\mu} k_{\nu} - k^2 \delta_{\mu\nu} \right) \tag{4.11}$$

in momentum space corresponds to a term of the form $F_{\mu\nu}\partial^{2n}F^{\mu\nu}$ in the effective action. This is because in momentum space

$$F_{\mu\nu}(-k)F^{\mu\nu}(k) \longrightarrow -(ik_{\mu}A_{\nu}(-k) - ik_{\nu}A_{\mu}(-k))(ik^{\mu}A^{\nu}(k) - ik^{\nu}A^{\mu}(k)) = -2(k_{\mu}k_{\nu} - k^{2}g_{\mu\nu})A^{\mu}(-k)A^{\nu}(k) .$$
(4.12)

Since the field strength $F_{\mu\nu}$ is gauge invariant, the one-loop calculation is consequently entirely consistent with gauge invariance. Taking the limit $\epsilon \to 0$, (4.10) includes the factor

$$\int_{0}^{1} dx \, (1-2x)^{2} \left[\frac{1}{8\pi^{2}\epsilon} - \frac{\gamma_{E}}{16\pi^{2}} - \frac{1}{16\pi^{2}} \log \frac{x(1-x)k^{2} + m^{2}}{4\pi\mu^{2}} \right] \,. \tag{4.13}$$

In the $\overline{\mathrm{MS}}$ scheme the required counter-term is

$$\mathscr{L}_{\rm ct} = -\frac{1}{12} \Big[-\frac{1}{8\pi^2 \epsilon} + \frac{\gamma_E - \log 4\pi}{16\pi^2} \Big] F_{\mu\nu} F^{\mu\nu}$$
(4.14)

where we used

$$\int_0^1 (1 - 2x)^2 = \frac{1}{3} . \tag{4.15}$$

Being careful with the overall normalization, gives the one-loop effective action for the photon in Minkowski space in the $\overline{\text{MS}}$ scheme:

$$S_{\text{eff}}[A_{\mu}] = -\frac{1}{4} \int d^d k \left[\frac{1}{e^2} - \frac{1}{16\pi^2} \int_0^1 dx \, (1 - 2x)^2 \log \frac{m^2 - x(1 - x)k^2}{\mu^2} \right] F_{\mu\nu}(-k) F^{\mu\nu}(k) \,. \tag{4.16}$$

The one-loop beta function in the $\overline{\text{MS}}$ scheme can then be deduced by demanding that S_{eff} satisfies an RG equation of the form (1.3) but with no wavefunction renormalization:

$$S_{\text{eff}}[A_{\nu};\mu,e(\mu)] = S_{\text{eff}}[A_{\nu};\mu',e(\mu')] . \qquad (4.17)$$

This yields the beta function

$$\mu \frac{de}{d\mu} = \frac{e^3}{16\pi^2} \int_0^1 dx \, (1 - 2x)^2 = \frac{e^3}{48\pi^2} \,. \tag{4.18}$$

It is interesting to repeat the calculation for a fermion. A Weyl fermion² couples to a gauge field via the kinetic term

$$\mathscr{L}_{\rm kin} = i\bar{\psi}^{\dot{\alpha}}\bar{\sigma}^{\mu}{}_{\dot{\alpha}\alpha}D_{\mu}\psi^{\alpha} . \tag{4.19}$$

In this case the fermion contributes to the vacuum polarization via the diagrams



If one follows through the derivation, the only difference in the final result amounts to replacement of $(1-2x)^2 \rightarrow 4x(1-x)$ in $(4.16)^3$.

Solving for the running coupling

$$\frac{1}{e(\mu)^2} = C - \frac{1}{24\pi^2} \log \mu , \quad \text{or} \ e(\mu)^2 = \frac{24\pi^2}{\log \Lambda/\mu} .$$
(4.20)

Notice that e, like the coupling g_4 in scalar field theory, is irrelevant at the Gaussian fixed point. So $e(\mu)$ becomes smaller in the IR but has a Landau pole in the UV—at least within the one-loop approximation. Hence QED, like scalar QFT, does not seem to have a continuum limit in d = 4. Of course this does not rule out a more complicated "UV completion" a issue that we will return to later.

There is tricky issue with regularization schemes like $\overline{\text{MS}}$ which do not involve setting conditions at a particular momentum, or energy, scale. If we compare the beta functions in (2.16), and (3.18) then the former exhibit manifest decoupling meaning that if we start with some cut-off $\mu > m$ and flow down towards the IR then when $\mu < m$ the RG flows are suppressed. This makes perfect physical sense: a particle of mass m only has effects when the energy scale is greater than. However, in the MS scheme (3.18) (or in the case of QED (4.18)) there does not appear to be any decoupling at all, the coupling is still running when $\mu \ll m$. While there is nothing wrong with this in principle, it is not very physical; in perturbation theory powers of $e(\mu) \to 0$ as $\mu \to 0$ will be compensated by powers of $\log \mu \to -\infty$ as in (4.16). Although, we shouldn't make the mistake of thinking of the couplings in the action as physical couplings, we can introduce a more physically motivated coupling in a momentum-independent regularization schemes like $\overline{\text{MS}}$ by decoupling the massive particle by hand when the RG scale μ crosses the threshold at $\mu = m$. The recipe is as follows: two theories are written down, one including the particle valid for $\mu \ge m$ and one without the field for $\mu \le m$. At $\mu = m$ physical quantities in the the two theories are matched. For example, in scalar QED, for $\mu \ge m$ the coupling $e(\mu)$ runs as in (4.18). At $\mu = m$ the electron is removed from the theory to leave the photon by itself. This is a non-interacting theory and the couplings does not run, rather it is frozen at the value e(m). The nasty log μ factors disappear from perturbation theory to be left with a nice power series in e(m) which is a useful series if e(m) is small enough, which in QED is the case since $\frac{e(m_e)^2}{4\pi} = \frac{1}{137}$.

Hence, in this more general philosophy as one flows into the IR, particles are decoupled and removed from the theory as the RG scale μ passes the appropriate mass thresholds. In a mass-dependent scheme decoupling is automatic. For example, if instead of performing minimal subtraction we rather fix the value of the effective coupling in (4.16) at the (Euclidean) momentum scale $k^2 = \tilde{\mu}^2$ (see note 4 of the last lecture). This requires a counter-term

$$\mathscr{L}_{\rm ct} = -\frac{1}{4} \int_0^1 dx \, (1-2x)^2 \Big[\frac{1}{8\pi^2 \epsilon} - \frac{1}{16\pi^2} \log \frac{x(1-x)\tilde{\mu}^2 + m^2}{\mu^2} \Big] F_{\mu\nu} F^{\mu\nu} \,. \tag{4.21}$$

We now think of $\tilde{\mu}$ as the RG scale and the beta-function follows as

$$\tilde{\mu}\frac{de}{d\tilde{\mu}} = \frac{e^3}{16\pi^2} \int_0^1 dx \, (1-2x)^2 \frac{\tilde{\mu}^2 x (1-x)}{m^2 + \tilde{\mu}^2 x (1-x)} \tag{4.22}$$

which does indeed vanish when $\tilde{\mu} \ll m$ —unlike (4.18). A comparison between the 2 running couplings in the form of $e^{-3}\mu de/d\mu$ (the $\overline{\text{MS}}$ one dashed) is shown below



Non-abelian gauge theories

Now we turn to the generalization of the RG flow of the gauge coupling to non-abelian gauge theories. The Lagrangian for a non-abelian gauge field coupled to a massive scalar field transforming in a representation r of the gauge group G has the form

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\,\mu\nu} + \left| D_\mu \phi_i \right|^2 - m^2 |\phi_i|^2 , \qquad (4.23)$$

where

$$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + f^{abc}A^{b}_{\mu}A^{c}_{\nu} , \qquad (4.24)$$

and

$$D_{\mu}\phi_i = \partial_{\mu}\phi_i + iA^a_{\mu}T^a_{ij}\phi_j , \qquad (4.25)$$

where f^{abc} are the structure constants of the gauge group G and T^a_{ij} are the generators of a representation r of G, so as matrices

$$[T^a, T^b] = i f^{abc} T^c . (4.26)$$

The RG flow of the gauge coupling g can be determined in similar way to QED. Integrating out the matter field produces a contribution which is exactly (4.18) but multiplied by the group theory factor C(r) where $C(r)\delta_{ab} = \text{Tr} T^a T^b$, for example, for gauge group SU(N), for the adjoint representation r = G, C(G) = N and $C(\text{fund}) = \frac{1}{2}$. However, the new feature is that there are contributions from the gauge field itself because of its self-interactions. The relevant graphs are⁴



The resulting expression for the one-loop beta-function including the contributions from a series of scalar fields, as above, and, in addition, Weyl fermions in representations r_f , all with a mass $m < \mu$, is

$$\mu \frac{dg}{d\mu} = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3} C(G) - \frac{1}{3} \sum_{\text{scalars } f} C(r_f) - \frac{2}{3} \sum_{\text{fermions } f} C(r_f) \right) \,. \tag{4.27}$$

For example, for G = SU(N) and N_s scalars and N_f Weyl fermions in the fundamental N representation plus the anti-fundamental \bar{N}

$$\mu \frac{dg}{d\mu} = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3}N - \frac{1}{3}N_s - \frac{2}{3}N_f\right) \,. \tag{4.28}$$

While if all fields are in the adjoint representation

$$\mu \frac{dg}{d\mu} = -\frac{Ng^3}{(4\pi)^2} \left(\frac{11}{3} - \frac{1}{3}N_s - \frac{2}{3}N_f\right) \,. \tag{4.29}$$

What is remarkable is that the contribution from the self interactions of the gauge field have an opposite sign from the matter fields. If we write the one-loop beta function as $\mu dg/d\mu = -bg^3$ then the resulting behaviour depends crucially on the sign of b. Solving the one-loop beta function equation $\mu dg/d\mu = -bg^3$ gives

$$g(\mu)^2 = \frac{1}{C + b \log \mu} = \frac{1}{b \log(\mu/\Lambda)}$$
, (4.30)

where we have written the integration constant as a dimensionful parameter Λ . Once again we have an example of *dimensional transmutation* where the dimensionless gauge coupling has turned into a dimensionfull scale. When there is sufficient matter so that b < 0, the situation will be as in QED: we must have $\mu < \Lambda$ and $g(\mu)$ will decrease with decreasing μ , the theory is IR free, and so the coupling is marginally irrelevant at the Gaussian fixed point at g = 0 and apparently no continuum limit exists. In this case, Λ is the position of the Landau pole in the UV as for ϕ^4 theory.

On the contrary, as long as the amount of matter is not too large so that b > 0, we must have $\mu > \Lambda$ so that the coupling $g(\mu)$ increases with decreasing μ . In this case the coupling is marginally relevant at the Gaussian fixed point and a non-trivial interacting continuum limit of the theory exists since there is a renormalized trajectory that flows out of the Gaussian fixed point. The situation is rather different from scalar fields in d < 4 because the UV limit of the theory is non-interacting, a situation known as asymptotic freedom. In this case, Λ signals the scale at which the coupling becomes large and perturbation theory breaks down. It is at these scales that confinement sets in. It is important to realize that Λ is a physically measurable scale.

The Higgs mechanism

There is another mechanism which affects the running of the gauge coupling. This is the phenomenon of *spontaneous symmetry breaking* or the *Higgs effect* which is driven by VEVs for the scalar fields. In general, a VEV ϕ^0 will not be invariant under gauge transformations. In other words, we can split the generators of the gauge group into 2 sets $\mathcal{H} \cup \mathcal{B}$:

$$T^a_{ij}\phi^0_j = 0 \qquad a \in \mathcal{H} , \quad T^a_{ij}\phi^0_j \neq 0 \qquad a \in \mathcal{B} .$$

$$(4.31)$$

The first set are the unbroken symmetries which generate a subgroup $H \subset G$. The second set are the broken symmetries. The gauge fields corresponding to the broken symmetries gain a mass which is determined by the kinetic term of the scalar:

$$D_{\mu}\phi^{\dagger}D^{\mu}\phi \longrightarrow \sum_{a,b\in\mathcal{B}} \left(\phi^{0\dagger}T^{a}T^{b}\phi^{0}\right)A^{a}_{\mu}A^{b\mu} .$$

$$(4.32)$$

For example, consider a scalar field in the N of SU(N). Up to gauge transformation we can suppose the VEV takes the form

$$\phi_i^0 = v\delta_{i1} \ . \tag{4.33}$$

If we represent the gauge field as a traceless $N \times N$ matrix, then

$$A^{a}_{\mu}T^{a}_{ij}\phi^{0}_{j} = A_{\mu ij}\phi^{0}_{j} = vA_{\mu i1} .$$
(4.34)

Hence, the components $A_{\mu i1} = A^*_{\mu 1i}$ gain a mass v and the gauge group is broken from SU(N) to SU(N-1). It should not be surprising that at the scale $\mu = v$ the beta function of the gauge coupling changes discontinuously (in a momentum independent scheme like $\overline{\text{MS}}$) from that appropriate to a gauge group G to one associated to a gauge group H.

UV Completion

We have seen that an essential requirement in order to take a continuum limit in a QFT is that the theory in the IR lies on a renormalized trajectory, that is the RG flow out of some UV fixed point. For two theories that we have met, ϕ^4 scalar QFT and QED, both in d = 4, there is no renormalized trajectory issuing from a fixed point and so when we try to take the continuum limit there is a Landau pole: as we run the RG backwards the coupling diverges to ∞ at some finite μ .

If this was the last word on continuum limits, then this would seem to be rather disastrous for the standard model which includes QED and the Higgs scalars. However, we have already seen that as the RG scale passes a mass of particle into the regime $\mu < m$, the particle is essentially redundant as far as the physics is concerned and in RG schemes like MS is actually removed by hand from the theory (although it is important to remember that it leaves its mark in the form of irrelevant operators suppressed by powers of 1/m which in principle will leave observable effects.) So in our theories with Landau poles, it may be that there is a good UV*completion* which requires new particles and fields as we flow into the UV. For example, QED, along with the other gauge interactions of the standard model with group $U(1) \times SU(2) \times SU(3)$ can be embedded in a larger group, the simplest being SU(5). This is idea of grand unification. The SU(5) is then broken to the standard model, including QED, by certain Higgs fields at certain energy scales such that the standard model is the effective theory at low energies. However, in the UV the theory is asymptotically free and so there exists a continuum limit (at least if we ignore the Landau pole in the Higgs sector). The Higgs sector of the standard model can also be given a consistent UV completion in models where it is the bound state of fermions, for example.

Notes

1 Of course this will only be true if the cut-off scheme itself does not break gauge invariance. For instance the sharp momentum cut-off is not consistent with gauge invariance because gauge transformations mix up modes of different frequency, *i.e.* they do not preserve the split (2.7).

2 A Weyl fermion ψ_{α} , $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$, is a 2-component complex Grassmann quantity. In Minkowski space its conjugate is denoted $\bar{\psi}^{\dot{\alpha}} = (\psi_{\alpha})^{\dagger}$. In Euclidean space ψ_{α} and $\bar{\psi}^{\dot{\alpha}}$ are treated as independent real objects. A Weyl fermion can also be represented in 4-component Dirac fermion

language as a Majorana fermion $\Psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$. A Dirac fermion, like the electron-positron field is then made up of two Weyl fermions $\Psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}$. In this 2-component notation the σ matrices can be taken as

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = \left(\mathbf{1}, \sigma^{i}\right) , \qquad \bar{\sigma}^{\mu\dot{\alpha}\alpha} = \left(\mathbf{1}, -\sigma^{i}\right) . \tag{4.35}$$

3 In general if we have a set of complex scalars and Weyl fermions with integer charges q_i meaning that covariant derivatives involve $D_{\mu} = \partial_{\mu} + ieq_i A_{\mu}$, or $\partial_{\mu} + iq_i A_{\mu}$ after the rescaling $A_{\mu} \to A_{\mu}/e$, then the one-loop beta function in the $\overline{\text{MS}}$ scheme is

$$\mu \frac{de}{d\mu} = \frac{1}{48\pi^2} \sum_{\text{scalars}} q_i^2 + \frac{1}{12\pi^2} \sum_{\text{fermions}} q_i^2 .$$
(4.36)

4 Depending on how the gauge is fixed, there will usually be a contribution from the gauge-fixing ghost fields. The complete calculation can be found in Peskin and Schroeder p533 onwards.

5 RG and Supersymmetry

QFTs with supersymmetry (SUSY) have some remarkable RG properties. From our point-of-view, one can prove the existence of many non-trivial fixed points of the RG. Our discussion of SUSY theories will be geared towards describing some of these fixed points and the associated RG flows and so will be very brief about other aspects of SUSY theories. In particular, we shall use component fields rather than introduce the whole paraphernalia of superspace and superfields and will be rather brief when discussing the global R-symmetries of SUSY theories.

In a SUSY theory, fields are collected into SUSY multiplets. In d = 4 (which we will stick to from now on) there are 2 basic multiplets: (i) a chiral multiplet consisting of a complex scalar, a Weyl fermion and a complex auxiliary scalar field

$$\Phi = (\phi, \psi_{\alpha}, F) , \qquad (5.1)$$

(ii) a vector multiplet consisting of a gauge field, an adjoint-valued Weyl fermion and an adjoint-valued auxiliary real scalar field

$$V = (A_{\mu}, \lambda_{\alpha}, D) . \tag{5.2}$$

Theories of chiral multiplets: Wess-Zumino models

Wess-Zumino models are constructed from chiral multiplets Φ_i with a basic SUSY kinetic term of the form¹

$$\mathscr{L}_{\rm kin} = |\partial_{\mu}\phi_i|^2 + i\bar{\psi}_i\bar{\sigma}^{\mu}\partial_{\mu}\psi_i + |F_i|^2 , \qquad (5.3)$$

note that the auxiliary fields F_i has trivial kinetic terms and their rôle in the theory is to simplify the SUSY structure. The interactions are determined by a function $W(\phi_i)$, the superpotential,

$$\mathscr{L}_{\text{int}} = F_i \frac{\partial W}{\partial \phi_i} + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_i \psi_j + \text{c.c.}$$
(5.4)

(c.c.=complex conjugate.) Notice that the F_i can trivially be "integrated out" by their equation-ofmotion

$$F_i = \frac{\partial W(\phi^*)}{\partial \phi_i^*} , \qquad (5.5)$$

to give a net potential on the scalar fields:

$$V(\phi_i, \phi_i^*) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 .$$
(5.6)

The theory is invariant under the infinitesimal SUSY transformations Grassmann parameter ξ_{α}

$$\delta\phi_i = \xi\psi_i ,$$

$$\delta\psi_i = i\sigma^{\mu}\bar{\xi}\partial_{\mu}\phi_i + \xi F_i ,$$

$$\delta F_i = i\bar{\xi}\bar{\sigma}^{\mu}\partial_{\mu}\psi_i .$$
(5.7)

There is an important distinction between the kinetic and interaction terms: \mathscr{L}_{kin} are "D terms", meaning functions of the fields and their complex conjugates, while \mathscr{L}_{int} are "F terms", consisting of the sum of a part which is holomorphic in the fields and the couplings and the complex conjugate term which has all quantities replaced by the corresponding anti-holomorphic ones. In particular, the superpotential $W(\phi_i)$ is a holomorphic function of the ϕ_i and also the couplings g_n , *i.e.* $W(\phi_i)$ does not depend on ϕ_i^* or g_n^* . It is this underlying holomorphic structure of F-terms that plays the pivotal rôle in proving exact RG statements in SUSY theories.

For example, for a ϕ^4 type model the superpotential has the form

$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}\lambda\phi^3 , \qquad (5.8)$$

and note that the coupling m and λ are in general complex. In this case, after integrating out the auxiliary fields F_i , the interaction terms are in this case

$$\mathcal{L}_{\text{int}} = -\left|m\phi + \lambda\phi^2\right|^2 - m\psi\psi - m^*\bar{\psi}\bar{\psi} - 2\lambda\phi\psi\psi - 2\lambda\phi^*\bar{\psi}\bar{\psi} .$$
(5.9)

The final 2 terms here are Yukawa interactions between the fermions and scalars.

Renormalization of the F-terms

From the point-of-view of RG, the key fact is that couplings in the superpotential do not change when we change the RG scale μ .² This does *not* mean that the couplings in the superpotential do not have any RG flow because there can be be non-trivial wavefunction renormalization, in addition to the canonical scaling coming from the fact that we always think of RG flow of dimensionless couplings. In fact, for a term $W \sim \mu^{3-n} \lambda \phi^n$, the RG equation becomes

$$W(Z(\mu)^{1/2}\phi;\mu,\lambda(\mu)) = W(Z(\mu')^{1/2}\phi;\mu',\lambda(\mu'))$$
(5.10)

and so under an RG flow

$$\mu^{3-n} Z(\mu)^{-n/2} \lambda(\mu) = \mu'^{3-n} Z(\mu')^{-n/2} \lambda(\mu') , \qquad (5.11)$$

from which we extract the beta-function³

$$\mu \frac{d\lambda}{d\mu} = (-3 + n + n\gamma)\lambda , \qquad (5.12)$$

where γ is the common anomalous dimension of all fields in the chiral multiplet (SUSY ensures that all fields in a multiplet have the same wavefunction renormalization). Another way to state this, is that the scaling dimension of the *composite operator* ϕ^n is just equal to the sum of the scaling dimension of the individual operators ϕ :⁴

$$\Delta_{\phi^n} = n(1+\gamma) = n\Delta_\phi . \tag{5.13}$$

On the contrary, SUSY does not constraint the running of D-term couplings.

The fact that the superpotential is only renormalized by wavefunction renormalization has a very important application. The vacua of a theory are determined by minimizing the effective potential. This latter quantity is, as argued previously, the potential in the Wilsonian effective action in the limit that $\mu \to 0$. In a SUSY theory, the potential, is given by (5.6) and so the absolute minima of V correspond to the critical points of the superpotential, or "F-flatness" condition

$$F_i^* = -\frac{\partial W(\phi_j)}{\partial \phi_i} = 0.$$
(5.14)

Such vacua can be shown to preserve SUSY, whereas if the minima have V > 0 then SUSY is spontaneously broken. Now we come to the important bit, as we decrease the cut-off $\mu \to 0$, if we have a solution ϕ_i of (5.14) at the original cut-off, then we still have a solution, differing only by wavefunction renormalization, $Z_i(\mu)^{1/2}\phi_i$.

For example, the model with a bare superpotential (5.8) has SUSY vacua at $\phi = 0$ and $\phi = -m/\lambda$ and by the argument above these vacua will persist, only shifted by wavefunction renormalization, once the quantum corrections are taken into account. It is also possible to cook-up models which do not have any SUSY vacua.

Vacuum Moduli Spaces

In a non-SUSY QFT, vacua of the theory are generally discrete points because RG flow generically lifts any flat directions of the potential. In a SUSY theory, however, RG flow cannot lift a degenerate space of SUSY vacua. So in many cases SUSY theories have non-trivial Vacuum Moduli Spaces \mathfrak{M} whose points are labelled by the VEVs of the scalar fields in the theory. The space \mathfrak{M} is in many cases not a manifold rather it is a series of manifolds, or "branches", often joined along subspaces of lower dimension.

As an example consider a model with 3 chiral multiplets X, Y and Z with a superpotential $W = \lambda xyz$. The SUSY vacua are determined by the equations xy = yz = zx = 0 and so there are 3 branches (i) y = z = 0, x arbitrary, (ii) z = x = 0, y arbitrary, and (iii) x = y = 0, z arbitrary. Here, x, *etc*, is short for the VEV of the scalar field component of X. These three branches are joined at the point x = y = z = 0. The structure of \mathfrak{M} is schematically



SUSY gauge theories

A SUSY gauge theory is constructed from a kinetic term which, unlike the chiral multiplet, is an F-term

$$\mathscr{L}_{\rm kin}(V) = \left(\frac{1}{2g^2} + \frac{\theta}{16\pi^2 i}\right) \mathscr{L}(V) + {\rm c.c.} ,$$

$$\mathscr{L}(V) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{i}{8} \epsilon_{\mu\nu\rho\sigma} F^{a\mu\nu} F^{a\rho\sigma} - i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^{a2} .$$
(5.15)

Notice that the gauge coupling is naturally combined with the θ angle⁵ to form a holomorphic coupling which is conventionally written as

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \ . \tag{5.16}$$

If we couple a vector superfield to a chiral superfield transforming in some representation r, with generators T_{ij}^a , of the gauge group, then SUSY requires minimal coupling: all derivatives are replaced by covariant derivatives. In addition, there are the extra interactions

$$\mathscr{L}_{\text{int}} = i\sqrt{2}T^a_{ij} \left(\phi^*_i \psi_j \lambda^a - \phi_i \bar{\psi}_j \bar{\lambda}^a\right) + D^a T^a_{ij} \phi^*_i \phi_j \ . \tag{5.17}$$

Notice that once the auxiliary fields D^a and F_i are integrated-out, the scalar fields have a potential

$$V(\phi_i, \phi_i^*) = \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{1}{2} \sum_a \left(\phi_i^* T_{ij}^a \phi_j \right)^2 , \qquad (5.18)$$

which generalizes (5.6). Henceforth, we will think of ϕ as a vector and leave the indices implicit. If we have several matter fields transforming in representations r_f then we will denote them with a "flavour index" Φ_f . For instance, SUSY QCD with gauge group SU(N) is defined as the theory with N_f chiral multiplets in the N representation, conventionally denoted $Q_f = (q_f, \psi_f, F_f)$, and N_f chiral multiplets in the \bar{N} representation, denoted $\tilde{Q}_f = (\tilde{q}_f, \tilde{\psi}_f, \tilde{F}_f)$.

Since the gauge coupling is a secretly a holomorphic quantity one wonders whether it has any non-trivial renormalization. The reason is that the perturbative expansion is in g and not τ , but renormalization must preserve the holomorphic structure hence there should be no RG flow of τ and hence g. Actually, this argument is a bit too quick, because one-loop running is consistent with holomorphy, since for the theory without matter fields, using (4.27) with one fermion in the adjoint representation, we have

$$\mu \frac{dg}{d\mu} = -\frac{3}{16\pi^2} C(G) g^3 \qquad \Longrightarrow \qquad \mu \frac{d\tau}{d\mu} = \frac{3i}{2\pi} C(G) \ . \tag{5.19}$$

which is consistent because θ does not run. So the beta function of g is exact at the one-loop level!

The re-scaling anomaly

Once we add chiral multiplets coupled to the vector multiplet the beta function of g will get nontrivial contributions for two reasons. The first effect is simple: the one-loop coefficient receives additional contributions from the fields of the chiral multiplets. From (4.27), and taking account the field content of the vector and chiral multiplets, we simply have to perform the replacement

$$3C(G) \longrightarrow 3C(G) - \sum_{f} C(r_f)$$
 (5.20)

The second contribution is more subtle. As we have learned under the RG transformation, as the cut-off μ is lowered the kinetic terms changes by the wavefunction renormalization factor:

$$|\partial_{\mu}\phi|^2 \longrightarrow Z |\partial_{\mu}\phi|^2 . \tag{5.21}$$

At this point, we have to perform the re-scaling $\phi \to Z^{-1/2}\phi$ in order to return the kinetic term to its canonical form. In principle, when we perform this re-scaling we should take into account the Jacobian coming from the measure of the functional integral. With a sharp momentum cut-off on the modes this gives a Jacobian

$$\operatorname{Jac} = Z^{\# \text{ degrees-of-freedom}} = \exp\left[\log Z \int d^d x \int_{|p| \le \mu} \frac{d^d p}{(2\pi)^d}\right] = \exp\left[\log Z \,\mu^d \,\frac{\operatorname{Vol} S^{d-1} \int d^d x}{(2\pi)^d}\right] \,. \tag{5.22}$$

This is divergent but still just a constant factor that we can safely ignore. However, when ϕ is coupled non-trivially to a gauge field, the Jacobian will depend on the background gauge field. The reason is in order to preserve gauge invariance the cut-off must somehow be compatible with the covariant derivative $D_{\mu} = \partial_{\mu} + iA_{\mu}$ and this means that the whole cut-off procedure must involve the background gauge field in a non-trivial way. The proper way to do this is to put the cut-off μ^2 on the eigenvalues of the covariant Laplace equation (in Euclidean space)

$$-D_{\mu}^{2}\phi = \lambda\phi . (5.23)$$

It is technically quite complicated to calculate the Jacobian this way; however, we can avoid this by making an intelligent use of the chiral anomaly.

The Chiral Anomaly

A chiral transformation takes

$$\psi \to e^{i\alpha}\psi$$
, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$. (5.24)

The action is clearly invariant under this transformation, however, the regularized measure $[d\psi][d\bar{\psi}]$ is not. The point is that the cut-off procedure breaks the symmetry in the presence of a background gauge field. Under the transformation the fermion part of the measure picks up a non-trivial Jacobian

$$[e^{i\alpha}d\psi][e^{-i\alpha}d\bar{\psi}] = [d\psi][d\bar{\psi}]\exp\left[-\frac{i\alpha C(r)}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\rho\sigma}F^{a\mu\nu}F^{a\rho\sigma}\right].$$
 (5.25)

What is remarkable about the chiral anomaly is that this result is exact to all orders in perturbation theory.

We can now use the chiral anomaly to calculate the re-scaling anomaly, or Jacobian, for a chiral multiplet by exploiting holomorphy and SUSY. If one compares the right-hand side of (5.25) with the SUSY kinetic term (5.15), taking into account that the holomorphic and anti-holomorphic contributions to \mathscr{L}_{kin} are separately invariant under SUSY, it is clear that (5.25) can be written in a way manifestly invariant under SUSY; namely

$$\operatorname{Jac} = \exp\left\{\frac{\alpha C(r)}{8\pi^2} \int d^4x \left[\mathscr{L}(V) - \mathscr{L}^*(V)\right]\right\} = \exp\left\{-\frac{iC(r)}{8\pi^2} \int d^4x \left[\left(i\alpha \mathscr{L}(V)\right) + \left(i\alpha \mathscr{L}(V)\right)^*\right]\right\}.$$
(5.26)

Most of the terms here cancel to leave only the right-hand side of (5.25). However, we can now exploit holomorphy to generalize the chiral transformation to an arbitrary holomorphic re-scaling of the whole chiral multiplet:

$$\phi \to Z^{-1/2}\phi$$
, $\psi \to Z^{-1/2}\psi$, $F \to Z^{-1/2}F$. (5.27)

The resulting Jacobian is then simply the right-hand side of (5.26) with $e^{i\alpha}$ replaced by $Z^{-1/2}$:

$$\operatorname{Jac} = \exp\left\{\frac{iC(r)}{16\pi^2} \int d^4x \left[\left(\log Z \,\mathscr{L}(V)\right) + \left(\log Z \,\mathscr{L}(V)\right)^*\right]\right\}.$$
(5.28)

Now this result is also valid, by analytic continuation, for a field wavefunction renormalization for which Z is real. Hence, we can calculate the effect of wavefunction renormalization on the running of the gauge coupling. Under an infinitesimal RG transformation $Z = 1 - 2\gamma \delta \mu / \mu$ and so the Jacobian is of the form

$$\operatorname{Jac} = \exp\left\{-\frac{iC(r)\gamma}{8\pi^2}\int d^4x \left[\mathscr{L}(V) + \mathscr{L}^*(V)\right]\frac{\delta\mu}{\mu}\right\}$$
(5.29)

which corresponds to an additional contribution to the flow of the gauge coupling of

$$\delta\left(\frac{1}{g^2}\right) = \frac{C(r)\gamma}{4\pi^2} \cdot \frac{\delta\mu}{\mu} ,$$

$$\mu \frac{dg}{d\mu}\Big|_{\text{additional}} = -\frac{g^3}{8\pi^2} C(r)\gamma .$$
(5.30)

Hence, the exact beta function for a model with a series of chiral multiplets in representations r_f of the gauge group is

$$\mu \frac{dg}{d\mu} = -\frac{g^3}{16\pi^2} \left(3C(G) - \sum_f C(r_f)(1 - 2\gamma_f) \right) \,. \tag{5.31}$$

The NSVZ exact beta-function

The above is an exact result valid for the coupling that appears as $1/g^2$ in front of the gauge kinetic term. We would think that this is the same as the *canonical gauge coupling* g_c , the one that appears in the covariant derivatives $D_{\mu} = \partial_{\mu} + ig_c A_{\mu}$ because one can simply re-scale the gauge field $A_{\mu} \rightarrow g_c A_{\mu}$ (and all the other fields of the vector multiplet). But we have just learnt our lesson: these re-scalings cannot be done willy-nilly since one can expect a non-trivial Jacobian from measure and this means that $g \neq g_c$. Unfortunately the trick that worked for a chiral multiplet does not work in a simple way here since the chiral rotation of the gluino cannot be complexified to give the re-scaling anomaly of the vector multiplet, and so we just quote the result

$$\operatorname{Jac} = \exp\left\{\frac{iC(G)g^2}{16\pi^2}\log g_c \int d^4x \left(\mathscr{L}(V) + \mathscr{L}(V)^*\right)\frac{\delta\mu}{\mu}\right\}.$$
(5.32)

Notice that it has the same form as an adjoint-valued chiral multiplet but with the opposite sign and with $Z = g_c^2$. The condition that the kinetic term, after the re-scaling, has no coupling in front of the kinetic term gives the condition

$$g_c^2 \left(\frac{1}{g^2} - \frac{C(G)}{4\pi^2} \log g_c\right) = 1$$
(5.33)

which is the exact relation between the two definitions of the gauge coupling g and g_c . It follows that the beta function of the canonical coupling is

$$\mu \frac{dg_c}{d\mu} = \mu \frac{dg_c}{d\mu} = \mu \frac{dg}{d\mu} \cdot \frac{g_c^3}{g^3} \cdot \frac{1}{1 - C(G)g_c^2/8\pi^2}$$

= $-\frac{g_c^3}{16\pi^2} \cdot \frac{3C(G) - \sum_f C(r_f)(1 - 2\gamma_f)}{1 - C(G)g_c^2/8\pi^2}$. (5.34)

This is the famous Novikov-Shifman-Vainshtein-Zahakarov (NSVZ) beta function for SUSY gauge theories.

Vacuum structure

Searching for the vacua of a SUSY gauge theory is more complicated than for a Wess-Zumino model. The potential of a SUSY gauge theory (5.18) includes a contribution from the gauge sector which arises once the D field is integrated out. Clearly, as in the Wess-Zumino case, it is bounded below by 0 and for a SUSY vacuum we need V = 0 and so the conditions are

$$\frac{\partial W}{\partial \phi_f} = 0 ,$$

$$\sum_f \phi_f^{\dagger} T^a \phi_f = 0 ,$$
(5.35)

known as the F- and D-term equations, which have to solved moduli gauge transformations. Such minima correspond to vacua of the theory which preserve SUSY.

To see how this works, consider SUSY QED with chiral fields Q_1 and Q_2 of charge +1 and \tilde{Q}_1 and \tilde{Q}_2 of charge -1. Furthermore, let us suppose that the theory has no superpotential. Hence, only the D-term equation is non-trivial:

$$|q_1|^2 + |q_2|^2 - |\tilde{q}_1|^2 - |\tilde{q}_2|^2 = 0.$$
(5.36)

One way to solve this is to use gauge transformations to make q_1 real and then the D-term equation determines, say, q_1 . This breaks down when $q_1 = 0$, but then we can use q_2 instead, *etc.* All-in-all, modulo gauge transformation, we have a 3-complex dimensional space of solutions which defines the vacuum moduli space \mathfrak{M} . At any point in \mathfrak{M} , besides the origin, the VEV of the scalar fields breaks the gauge group and the photon gains a mass by the Higgs mechanism.

It is an important theorem, that any solution of the F- and D-term equations moduli gauge transformations is equivalent to a solution of the F-term equations *alone* modulo *complexified* gauge transformations.⁶ This theorem is useful because it survives quantum corrections. In particular, although the D-terms are renormalized non-trivially, we can ignore them if we are only interested in the SUSY vacua of theory. In the present case, complexified gauge transformations acts as

$$q_1 \to \lambda q_1 , \qquad q_2 \to \lambda q_2 , \qquad \tilde{q}_1 \to \lambda^{-1} \tilde{q}_1 , \qquad \tilde{q}_2 \to \lambda^{-1} \tilde{q}_2 .$$
 (5.37)

So for the example above, which has no F-term conditions, we can find \mathfrak{M} by finding a set if coordinates which are gauge invariant in a complexified sense: in this case $z_1 = q_1 \tilde{q}_1$, $z_2 = q_1 \tilde{q}_2$, $z_3 = q_2 \tilde{q}_1$ and $z_4 = q_2 \tilde{q}_2$. However, these coordinates are not all independent, a short-coming that can be remedied by imposing the condition

$$z_1 z_4 = z_2 z_3 . (5.38)$$

So the vacuum moduli space is an example of a *complex variety* defined by the above condition in \mathbb{C}^4 . In this case the complex variety is actually a *conifold* rather than a manifold since it is singular at $z_i = 0$ where the U(1) gauge symmetry is restored.

Notes

1 The kinetic term can be generalized to all the terms with two derivatives:

$$\mathscr{L}_{\rm kin} = \frac{\partial^2 K}{\partial \phi_i \partial \phi_j^*} \partial_\mu \phi_i \partial^\mu \phi_j^* + \text{terms involving } (\psi, F) , \qquad (5.39)$$

determined by a real function, $K(\phi, \phi^*)$, the Kähler potential.

2 The simple proof of this fact is to think of the holomorphic and anti-holomorphic fields and couplings as being temporarily independent. Then when we integrate out modes as we decrease μ the flow of the holomorphic couplings cannot depend on the anti-holomorphic couplings and so we can set the latter to 0. In this limit, both the scalar and fermion propagator have no components which can joint the legs of holomorphic fields. Hence, the graphs that contribute to the holomorphic fields simply vanish.

3 The 3 here is the canonical dimension of W and γ is the anomalous dimension of each field of the chiral multiplet which must be equal by SUSY. The generalization for a term $W \sim \mu^{3-p} \lambda \phi_{i_1} \cdots \phi_{i_n}$ is

$$\mu \frac{d\lambda}{d\mu} = \left(-3 + p + \sum_{i=1}^{n} \gamma_i\right) \lambda .$$
(5.40)

4 It is a key aspect of RG that scaling dimension of a general composite operator $\mathcal{O} = \mathcal{O}_1 \cdots \mathcal{O}_p$ is *not* usually the sum $\Delta_{\mathcal{O}} \neq \Delta_{\mathcal{O}_1} + \cdots + \Delta_{\mathcal{O}_p}$. For instance, for ϕ^4 theory around the Wilson-Fischer fixed point $\Delta_{\phi^4} \neq 2\Delta_{\phi^2}$.

5 The θ angle multiplies a term

$$\frac{1}{64\pi^2} \int d^4x \,\epsilon_{\mu\nu\rho\sigma} F^{a\mu\nu} F^{a\rho\sigma} \,, \qquad (5.41)$$

in the action. This integral is topological, in the sense that for any smooth gauge configuration it is equal to $2\pi k$, where k is an integer: the 2nd Chern Class of the gauge field. Furthermore, the theta term does not contribute to the classical equations-of-motion. In the quantum theory which involves the Feynman sum over configurations, θ is another coupling in the theory.

6 We present the proof in the abelian case G = U(1). Suppose we have a solution of the F-term equations q_f (with charges e_f). The key point is that since the superpotential only depends on the holomorphic fields and not the anti-holomorphic ones, it is actually invariant not just under gauge transformations $q_f \rightarrow e^{ie_f\theta}q_f$ but also complexified gauge transformations $q_f \rightarrow \eta^{e_f}q_f$, for arbitrary complex η . On the other hand, the D-term

$$\sum_{f} e_f |q_f|^2 \tag{5.42}$$

is not invariant under this complexified transformation. The strategy is then to use the complexified gauge transformation with η real, to find a solution of the D-term equation: $\check{q}_f = \eta^{e_f} q_f$:

$$\sum_{f} e_{f} |\check{q}_{f}|^{2} = \sum_{f} e_{f} \eta^{2e_{f}} |q_{f}|^{2} = \frac{1}{2} \frac{\partial}{\partial \eta} \sum_{f} \eta^{2e_{f}} |q_{f}|^{2} = 0 .$$
 (5.43)

It is always possible to find an η which does this because for the function $\sum_f \eta^{2e_f} |q_f|^2$ either (i) e_f all have the same sign, in which case $\eta \to 0$ or ∞ depending on the sign of the charges, or (ii) if some of the e_f have opposite sign $\sum_f \eta^{2e_f} |q_f|^2$ goes to ∞ for $\eta \to 0, \infty$, and hence has a minimum as a function of η for some finite $\eta > 0$.

The generalization to the non-abelian case is reasonably straightforward: see Wess and Bagger p.57-58.

6 More RG Structure of SUSY Theories and $\mathcal{N} = 4$

RG fixed points

The next issue we will address in our RG investigation of SUSY theories will be to show that there are many non-trivial fixed points of the RG in d = 4 supersymmetric gauge theories. These fixed point have a supersymmetric extension of conformal invariance and are consequently super-conformal field theories (SCFTs). The existence of these fixed points (actually manifolds) rests on the special properties that we have established for the renormalization of the superpotential (5.40) and for the gauge coupling (5.31) or (5.34).

First of all, consider a Wess-Zumino model (so only chiral multiplets). In that case, one can prove that all the anomalous dimensions $\gamma_i \geq 0$ (with equality for a free theory). Since, in order to have an interacting theory, we need terms cubic in the superpotential, it follows immediately that there cannot be non-trivial fixed points of RG:

$$\mu \frac{d\lambda}{d\mu} = (-3 + n + n\gamma)\lambda > 0.$$
(6.1)

Now we add vector multiplets and take the chiral multiplets in representations r_f of the gauge group. In that case, there is no positivity conditions on the anomalous dimensions of the chiral multiplets. In order to have a fixed point, for each coupling in the superpotential $\sim \lambda \phi_{f_1} \cdots \phi_{f_p}$, we need from (5.40)

$$\sum_{i=1}^{p} \gamma_{f_i} = 3 - p \tag{6.2}$$

and from (5.31)

$$3C(G) - \sum_{f} C(r_f) \left(1 - 2\gamma_f \right) = 0 .$$
(6.3)

If there are n such cubic couplings λ_i in the superpotential then it appears that there are n+1 equations for n+1 unknowns $\{\lambda_i, g\}$. Generically, therefore, if solutions exist they will be discrete and unlikely to be within the reach of perturbation theory. However, there are special situations when the set of n+1 equations are degenerate in which case there are spaces of fixed points which can extend into the perturbative regime.

For example, suppose there are 3 chiral multiplets in the adjoint representation with a superpotential which has sufficient symmetry to infer that all the anomalous dimensions are equal $\gamma_f \equiv \gamma$. In this case, (6.2) and (6.3) are satisfied if the superpotential is cubic in the fields and if the anomalous dimension

$$\gamma(\lambda_i, g) = 0 \tag{6.4}$$

which is a single condition on n + 1 couplings. Such theories are very special because they are actually *finite*.

Finite Theories

A theory is *finite* if there are no UV divergences in perturbation theory. In a SUSY theory, this means that the anomalous dimensions of all the chiral operators vanish and the beta function of the gauge coupling vanishes. The conditions are:

(i) $\gamma_f = 0$.

(ii) The superpotential must be cubic in the fields, in order that the RG flow of the couplings in the superpotential vanish: see (5.40).

(iii) $3C(G) = \sum_{f} C(r_f)$ in order that the RG flow of the gauge coupling vanishes.

Notice that not every finite theory is a CFT since conformal invariance could be broken by VEVs for scalar fields if there is a moduli space of vacua. However, in such cases conformal invariance would be recovered in the UV. Neither is every conformal field theory a finite theory since the condition to be at a fixed point does not require the anomalous dimensions of fields to vanish.

For example, if the gauge group is G = SU(N), there are 3 gauge invariant couplings one can write in the superpotential for 3 adjoint-valued fields ($N \times N$ traceless Hermitian matrices) which have enough symmetry to imply that all the anomalous dimensions are equal:

$$W = \operatorname{tr} \left(\lambda_1 \phi_1 \phi_2 \phi_3 + \lambda_2 \phi_1 \phi_3 \phi_2 + \frac{\lambda_3}{3} \sum_{f=1}^3 \phi_f^3 \right) \,. \tag{6.5}$$

Of course we need to check that the condition $\gamma(\lambda_1, \lambda_2, \lambda_3, g) = 0$ actually has solutions. The key to proving this is to establish that solutions exist in perturbation theory and then by continuity one can expect that solutions exist also at strong coupling. To one-loop it can be shown that the anomalous dimension is¹

$$\gamma = \frac{C(G)}{64\pi^2} \left(\sum_f |\lambda_f|^2 - 4g^2 \right) \,. \tag{6.6}$$

We won't give the proof of this, but it is easy to see the diagrams that contribute. For example, for the anomalous dimension of the scalar field, the following 1-loop graphs contribute (plus graphs involving the ghosts which we do not show)



So γ receives positive contributions from the chiral multiplets and negative contributions from the vector multiplet. To simplify the discussion suppose that all the couplings $\lambda_f \sim \lambda$. In this case, there is a RG-fixed line at weak coupling when $\lambda \sim g$. RG flow in the (λ, g) subspace is of the form



where the dotted line is a line of fixed points. So the couplings away from the fixed line in the (λ, g) subspace are irrelevant since the RG flow is into the fixed line. One would expect that the fixed line extends by continuity into the region of strong coupling as well.

Once we take all the couplings into account, there is actually a 6-dimensional space of SCFTs (this comes from the complex λ_f , f = 1, 2, 3 and the gauge coupling subject to one condition).

The maximally SUSY gauge theory

A special class of these finite theories corresponds to taking a SUSY gauge theory with 3 chiral multiplets in the adjoint representation with a superpotential of the form (6.5) with $\lambda_1 = -\lambda_2 = \sqrt{2}g$ and $\lambda_3 = 0$, *i.e.*¹

$$W = \sqrt{2}g\operatorname{Tr}\phi_1[\phi_2,\phi_3] . \tag{6.7}$$

Such theories have extended $\mathcal{N} = 4$ SUSY, the maximal amount of the SUSY in d = 4 for theories which do not contain gravity.

The theory has a large global SU(4) symmetry under which the 3 fermions from the chiral multiplets and the gluino transform as the 4, while the 3 complex scalars ϕ_i can be written as 6 real scalars that transform in the antisymmetric 6 (or the vector of $SO(6) \simeq SU(4)$). Such symmetries of SUSY theories for which the fermions and scalars transform differently are known as R-symmetries. It is no accident that SO(6) is the isometry group of an S^5 : see below. For later use, the potential of the theory has the form

$$V(\phi_i, \phi^*) = g^2 \sum_{ij=1}^{3} \text{Tr} \left(\phi_i \phi_j \phi_i^* \phi_j^* - \phi_i \phi_j \phi_j^* \phi_i^* \right) \,. \tag{6.8}$$

 $\mathcal{N} = 4$ are infinitely fascinating due to the amazing fact that they are equivalent to Type IIB string theory in 10-d spacetime on an $AdS_5 \times S^5$ background. The fact that what appears to be a humble 4d gauge theory can encode all the rich dynamics of a ten-dimensional (graviational) string theory is, I believe, one of the most amazing results in theoretical physics. We won't give a complete discussion of the AdS/CFT correspondence here since it will be discussed at length in other lectures at this school. However, since RG plays an important rôle, I will set the scene.

The gauge theory has 2 parameters g and N. In the dual string picture the string coupling $g_s = g^2/4\pi$ while the radius of the geometry in string units is $\sqrt{g^2N}$. This motivates the 't Hooft limit in which $N \to \infty$ with $\lambda = g^2N$ fixed. This is the so-called *planar* limit on the gauge theory side since only planar Feynman graphs survive, while the string side should correspond to free strings moving in an $AdS_5 \times S^5$ geometry. Perturbation theory on the gauge theory side requires λ small which is the limit on the string side where the geometry is highly curved. While strong coupling, large λ , in the gauge theory corresponds to strings moving on a weakly curved background.

Part of the "dictionary" is that single trace composite operators of the form

$$\mathcal{O}(x) = \operatorname{tr}\left(a_1 a_2 \cdots a_L\right) \,, \tag{6.9}$$

where the a_i are one of the fundamental fields (where A_{μ} appears through \mathcal{D}_{μ} in order to be gauge invariant) correspond to single string states. Moreover the scaling dimension of the operator equals the energy of the associated string state:

$$\mathscr{E}_{\text{string}} = \Delta_{\mathcal{O}}$$
 (6.10)

So perturbative calculations of the anomalous dimensions of operators in the gauge theory tells us directly about the spectrum of strings moving in a highly curved $AdS \times S^5$ geometry!

There have been some amazing developments in calculating these anomalous dimensions in the gauge theory and matching them to the energies of string states. Here, we will consider the problem of calculating the anomalous dimensions of single trace operators made up of just the two "letters" ϕ_1 and ϕ_2 to one-loop in planar perturbation theory (perturbation theory in λ with $N = \infty$ which suppresses completely non-planar contributions):

$$\mathcal{O} = \operatorname{tr}\left(\phi_1 \phi_1 \phi_2 \cdots\right) \tag{6.11}$$

If the operator has length L then the classical dimension is simply $d_{\mathcal{O}} = L$, so

$$\Delta_{\mathcal{O}} = L + \lambda \Delta_1 + \lambda^2 \Delta_2 + \cdots . \tag{6.12}$$

The problem is that the operators \mathcal{O}_i with a given number J_1 of ϕ_1 and J_2 of ϕ_2 , with $J_1 + J_2 = L$, all mix under RG and so we have the problem of diagonalizing a matrix.

One way to calculate the anomalous dimension of the class of such operators with fixed J_1 and J_2 , $\{\mathcal{O}_p\}$, is to add them to the action

$$S \longrightarrow S + \int d^d x \sum_p \mu^{4-L} g_p \mathcal{O}_p(x)$$
(6.13)

and then look at the flow of the g_p in the effective potential to linear order in g_p . We follow exactly the same background field method that we used earlier and treat the operator terms as new vertices in the action with couplings g_p . The flow of the couplings g_p can be deduced by writing down Feynman graphs with J_1 external ϕ_1 and J_2 external ϕ_2 lines with fluctuating fields on internal lines.

Since the anomalous dimension follows from

$$\mu \frac{dg_p}{d\mu} = (L-4)g_p + \gamma_{pq}g_q + \cdots , \qquad (6.14)$$

we only need look at graphs which use the vertices g_p once. At one-loop level, to begin with we have the graphs



The first graph has to be summed over all neighbouring pairs (but not non-neighbouring pairs, since these would be non-planar graphs suppressed by powers of 1/N), while the others are to summed over all L legs. The third involves a fermion loop and the fourth a scalar loop. The important point about these graphs is the they don't change the flavour of the legs of the vertex. In other words, whatever their contribution to the anomalous dimension is proportional to the identity in the space of (J_1, J_2) operators. We shall shortly argue that these contributions, which we write $C_1\mathbf{1}$, actually vanish, so $C_1 = 0$.

The remaining graphs involve using the quartic coupling in the scalar potential to tie two adjacent legs together. The potential (6.8) contains the terms

$$V = 2g^{2} \operatorname{tr} \left(\phi_{1} \phi_{2} \phi_{1}^{*} \phi_{2}^{*} - \phi_{1} \phi_{2} \phi_{2}^{*} \phi_{1}^{*} - \phi_{2} \phi_{1} \phi_{1}^{*} \phi_{2}^{*} + \phi_{2} \phi_{1} \phi_{2}^{*} \phi_{1}^{*} \right) + \cdots , \qquad (6.15)$$

and so we see immediately that these interactions can be used to form two additional one-loop graphs when two adjacent legs are different, either $\phi_1\phi_2$, as shown, or $\phi_2\phi_1$



The absolute contribution is simple to calculate, but even without explicit calculation it is easy to see from the potential that these contributions come with a relative -1.

Putting all this together, we have found that the two sets of one-loop contributions to the anomalous dimension "operator" can be written neatly as

$$\gamma = \sum_{\ell=1}^{L} \left\{ C_1 \mathbf{1}_{\ell} + C_2 \big(\mathbf{1}_{\ell} - \boldsymbol{P}_{\ell} \big) \right\}$$
(6.16)

where P_{ℓ} permutes the ℓ^{th} and $\ell + 1^{\text{th}}$ letters in the word and $\mathbf{1}_{\ell}$ is the identity:

$$\boldsymbol{P}_{\ell} \operatorname{tr} \left(\cdots \phi_{i_{\ell}} \phi_{i_{\ell+1}} \cdots \right) = \operatorname{tr} \left(\cdots \phi_{i_{\ell}+1} \phi_{i_{\ell}} \cdots \right)$$
(6.17)

and we identify $L + 1 \equiv 1$ due to the cyclicity of the trace.

Now we can pin down C_1 by using the fact that the operator tr ϕ_1^L is special because it is a so-called BPS operator and is consequently protected against quantum corrections; hence, $\Delta = L$ to all orders in the perturbative expansion. This means that $C_1 = 0$ a simple computation gives

$$C_2 = \frac{\lambda}{4\pi^2} . \tag{6.18}$$

Rather interestingly the resulting anomalous dimension operator γ up to some overall scaling is identical to the Hamiltonian of the so-called XXX spin chain, a quantum mechanical model of L spins taking values $|\uparrow\rangle \equiv \phi_1$ and $|\downarrow\rangle \equiv \phi_2$. Each operator of fixed length corresponds to a state of the spin chain:

$$\operatorname{tr}(\phi_1\phi_1\phi_2\phi_1\phi_2\phi_2\phi_1) \longleftrightarrow |\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle .$$
(6.19)

The problem of finding the anomalous dimensions and hence the spectrum of string states then is identical to the problem of finding the eigenstates of the spin chain, a problem that was solved by Bethe in 1931 by means of what we now call the *Bethe Ansatz* which reflects the fact that the problem is in the special class of integrable system. This observation is just the beginning of the fascinating story of $\mathcal{N} = 4$, integrability and the AdS/CFT correspondence.

Notes

1 The coupling here and in the following is generally the canonical gauge coupling, however, we won't distinguish the g and g_c for now on.

2 *F* is also complex, while *D* is Hermitian and for the fermions $\psi^{\dagger}_{\alpha} = \bar{\psi}^{\dot{\alpha}}$ and $\lambda^{\dagger}_{\alpha} = \bar{\lambda}^{\dot{\alpha}}$.

3 The perturbative expansion is really an expansion in g^2N . In our limit, however, $g^2 \sim 1/N^2$ and so higher orders in perturbation theory are suppressed by 1/N.