# Supersymmetry

Lecture 2

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

Alfonso V. Ramallo

# Spinors

Metric conventions $\longrightarrow$ $$g^{\mu\nu} = (1, -1, -1, -1)$$

## Dirac matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

## Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\sigma^\mu = (1, \sigma^i)$$
$$\bar{\sigma}^\mu = (1, -\sigma^i)$$

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \quad \Longrightarrow \quad \begin{aligned} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu &= 2g^{\mu\nu} \\ \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu &= 2g^{\mu\nu} \end{aligned}$$

## chirality matrix

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

## Weyl components

$$\Psi = \Psi_L + \Psi_R \qquad \Psi_R = \left(\frac{1+\gamma_5}{2}\right)\Psi \qquad \Psi_L = \left(\frac{1-\gamma_5}{2}\right)\Psi$$

$$\boxed{\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}} \qquad \Psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \qquad \Psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

**Dirac conjugate** → $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$ → $\bar{\Psi} = (\,\bar{\chi}^{\dot{\alpha}\,\dagger}\,,\,\psi_\alpha^\dagger\,)$

**Define** $\bar{\psi}_{\dot{\alpha}} = [\psi_\alpha]^\dagger \qquad \chi^\alpha = [\bar{\chi}^{\dot{\alpha}}]^\dagger$ → $\boxed{\bar{\Psi} = (\,\chi^\alpha\,,\,\bar{\psi}_{\dot{\alpha}}\,)}$

bar → dotted indices

undotted ⟷ (c. c.) dotted

# Raising and lowering indices

Define

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

They verify

$$\epsilon^{\gamma\alpha}\,\epsilon_{\alpha\lambda} = \delta^{\gamma}_{\lambda} \qquad \epsilon^{\dot{\gamma}\dot{\alpha}}\,\epsilon_{\dot{\alpha}\dot{\lambda}} = \delta^{\dot{\gamma}}_{\dot{\lambda}}$$

Then:

$$\chi_\alpha = \epsilon_{\alpha\beta}\,\chi^\beta \qquad \chi^\alpha = \epsilon^{\alpha\beta}\,\chi_\beta$$

$$\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\psi}_{\dot{\beta}} \qquad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\psi}^{\dot{\beta}}$$

Example:

$$\chi^1 = -\chi_2 \qquad \chi^2 = \chi_1$$

# Charge conjugation

$$\Psi^c = C\bar{\Psi}^T \qquad C\gamma_\mu^T C^{-1} = -\gamma_\mu$$

Take:

$$C = -i\gamma^0\gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$$

Then:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \Rightarrow \Psi^c = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$$

exchanges upper and lower components

Majorana spinors

$$\Psi_M = \Psi_M^c$$

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \chi \\ -i\sigma_2\,\chi^* \end{pmatrix}$$

# Lorentz covariance

## Transformation law

$$\Psi \rightarrow S\Psi \qquad S = e^{-\frac{i}{4}\omega_{\mu\nu}\Sigma^{\mu\nu}}$$

**with**

$$\frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} i\sigma^{\mu\nu} & 0 \\ 0 & i\bar{\sigma}^{\mu\nu} \end{pmatrix}$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta \qquad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}$$

**index structure**

$$(\sigma^\mu)_{\alpha\dot{\alpha}} \qquad (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$$

**Check:**

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\ \sigma^\mu_{\beta\dot{\beta}}$$

$\sigma^\mu$: dotted subindex $\rightarrow$ undotted subindex

$\bar{\sigma}^\mu$: undotted superindex $\rightarrow$ dotted superindex

## Define the 2x2 matrix

$$M = e^{\frac{1}{2}\,\omega_{\mu\nu}\,\sigma^{\mu\nu}} \Longrightarrow \left(M^{\dagger}\right)^{-1} = e^{\frac{1}{2}\,\omega_{\mu\nu}\,\bar{\sigma}^{\mu\nu}}$$

Then:

$$S = \begin{pmatrix} M & 0 \\ 0 & \left(M^{\dagger}\right)^{-1} \end{pmatrix}$$

## Transformation laws of two-component spinors:

$$\psi_\alpha \to [M\psi]_\alpha \qquad \psi^\alpha \to [\psi M^{-1}]^\alpha$$

$$\bar{\chi}^{\dot{\alpha}} \to \left[\left(M^{\dagger}\right)^{-1}\bar{\chi}\right]^{\dot{\alpha}} \qquad \bar{\chi}_{\dot{\alpha}} \to \left[\bar{\chi}\,M^{\dagger}\right]_{\dot{\alpha}}$$

$\psi^\alpha \chi_\alpha$ is a scalar $\qquad \bar{\psi}_{\dot{\alpha}}\,\bar{\chi}^{\dot{\alpha}}$ is a scalar

Denote

$$\chi\psi \equiv \chi^\alpha \psi_\alpha \longrightarrow \text{descending indices}$$

$$\bar\chi\bar\psi \equiv \bar\chi_{\dot\alpha} \bar\psi^{\dot\alpha} \longrightarrow \text{ascending indices}$$

Notice

$$\chi^\alpha \psi_\alpha = -\chi_\alpha \psi^\alpha \qquad \bar\chi_{\dot\alpha} \bar\psi^{\dot\alpha} = -\bar\chi^{\dot\alpha} \bar\psi_{\dot\alpha}$$

For anticonmmuting spinors

$$\chi\psi = \psi\chi \qquad \bar\chi\bar\psi = \bar\psi\bar\chi$$

From now on all spinors will be anticommuting

# SUSY algebra in two-component form

$$Q_r = \left( \begin{array}{c} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{array} \right)$$

Q Majorana

Poincare transformations:

$$[P^\mu, \bar{Q}_{\dot\alpha}] = [P^\mu, Q_\alpha] = 0$$

$$[M^{\mu\nu}, Q_\alpha] = -i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \qquad [M^{\mu\nu}, \bar{Q}^{\dot\alpha}] = -i(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}$$

## Anticommutators

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu$$

## Raising indices we get

$$\{Q_\alpha, Q^\beta\} = \{Q^\alpha, Q^\beta\} = 0$$

$$\{\bar{Q}_{\dot\alpha}, \bar{Q}^{\dot\beta}\} = \{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} = 0$$

$$\{\bar{Q}^{\dot\alpha}, Q^\beta\} = 2\,\bar\sigma^{\mu\ \dot\alpha\beta} P_\mu$$

# Representation of translations

$$\phi(x) \;\rightarrow\; e^{iy^\mu P_\mu}\,\phi(x)\,e^{-iy^\mu P_\mu} \;=\; \phi(x+y)$$

$$\phi(x) \;=\; e^{ix^\mu P_\mu}\,\phi(0)\,e^{-ix^\mu P_\mu}$$

## For y infinitesimal:

$$\phi(x+y) \;=\; \phi(x) \;+\; y^\mu\,\partial_\mu\,\phi(x) \;+\; \cdots$$

Thus, if $\delta\phi = \phi(x+y) - \phi(x)$

$$\delta\phi = -iy^\mu\,[\phi(x)\,,\,P_\mu\,] \;=\; y^\mu\,\frac{\partial}{\partial x^\mu}\,\phi(x)$$

$$\Downarrow$$

$$\boxed{\delta\phi = -iy^\mu\,[\phi(x)\,,\,P_\mu\,] \;=\; -iy^\mu\,\mathcal{P}_\mu\,\phi} \;\Rightarrow\; \boxed{\mathcal{P}_\mu \;\equiv\; i\,\frac{\partial}{\partial x^\mu}}$$

operator representing translations

# Superspace

Let us consider a Grassmann variable $\theta$:

$$\{\theta, \theta\} = 0 \implies \theta^2 = 0$$

For any function $F(\theta)$:

$$F(\theta) = \sum_{n=0}^{\infty} a_n \theta^n = a_0 + a_1 \theta$$

With two anticommuting variables $\theta_1$ and $\theta_2$:

$$F(\theta_\alpha) = a_0 + \theta_1 a_1 + \theta_2 a_2 + \theta_2 \theta_1 a_3$$

## We consider two pairs of Grassmann variables

$$\theta_\alpha \qquad \alpha = 1, 2$$

$$\bar{\theta}^{\dot{\alpha}} \qquad \dot{\alpha} = \dot{1}, \dot{2}$$

$$\{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta_\alpha, \bar{\theta}^{\dot{\beta}}\} = 0$$

Let $\eta^\alpha$ and $\bar{\eta}_{\dot{\alpha}}$ be anticommuting constant spinors

## The SUSY algebra in terms of commutators

$$[\eta Q, \bar{\eta}\bar{Q}] \equiv [\eta^\alpha Q_\alpha, \bar{\eta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}] = 2\,\eta\sigma^\mu\bar{\eta}\,P_\mu$$

$$[\eta Q, \eta' Q] = [\bar{\eta}\bar{Q}, \bar{\eta}'\bar{Q}] = 0$$

Let us consider a function in superspace $\mathcal{F}(x,\theta,\bar{\theta})$

## Operator of supertranslations

$$S(y,\eta,\bar{\eta}) = e^{i\left[y^{\mu}P_{\mu}+\eta\,Q+\bar{\eta}\bar{Q}\right]}$$

## Acts on functions as:

$$\mathcal{F}(x,\theta,\bar{\theta}) \;\rightarrow S(y,\eta,\bar{\eta})\,\mathcal{F}(x,\theta,\bar{\theta})\,S^{-1}(y,\eta,\bar{\eta})$$

## Translating $\mathcal{F}$ from the origin

$$\mathcal{F}(x,\theta,\bar{\theta}) \;\rightarrow S(y^{\mu},\eta,\bar{\eta})\,S(x^{\mu},\theta,\bar{\theta})\;\mathcal{F}(0,0,0)\;S^{-1}(x^{\mu},\theta,\bar{\theta})\;S^{-1}(y^{\mu},\eta,\bar{\eta})$$

# Using the Hausdorff formula

$$e^A e^B = e^{A+B+1/2[A,B]+\ldots}$$

## Multiplication law:

$$S(y^\mu, \eta, \bar{\eta})\, S(x^\mu, \theta, \bar{\theta}) = S(x^\mu + y^\mu + i\eta\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\eta}, \theta + \eta, \bar{\theta} + \bar{\eta})$$

$$\Downarrow$$

$$\mathcal{F}(x, \theta, \bar{\theta}) \longrightarrow \mathcal{F}(x^\mu + y^\mu + i\eta\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\eta}, \theta + \eta, \bar{\theta} + \bar{\eta})$$

## Supertranslations

$$x^\mu \longrightarrow x^\mu + y^\mu + i\eta\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\eta}$$

$$\theta \longrightarrow \theta + \eta \qquad \bar{\theta} \longrightarrow \bar{\theta} + \bar{\eta}$$

## Realization in terms of differential operators

For infinitesimal supertranslations

$$\delta_S \mathcal{F} = \left[ \eta^\alpha \frac{\partial}{\partial \theta^\alpha} + \bar{\eta}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + \left( y^\mu + i\eta\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\eta} \right) \frac{\partial}{\partial x^\mu} \right] \mathcal{F}$$

Using

$$\theta\sigma^\mu\bar{\eta} = -\bar{\eta}\bar{\sigma}^\mu \theta$$

We can write:

$$\delta_S \mathcal{F} = -iy^\mu P_\mu \mathcal{F} - i\eta^\alpha Q_\alpha \mathcal{F} - i\,\bar{\eta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}} \mathcal{F}$$

with:

$$Q_\alpha = i\left[ \frac{\partial}{\partial \theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\, \bar{\theta}^{\dot{\alpha}} \partial_\mu \right]$$

$$\bar{Q}^{\dot{\alpha}} = i\left[ \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \theta_\alpha \partial_\mu \right]$$

# Raising and lowering the indices:

$$Q^\alpha = -i\left[\frac{\partial}{\partial\theta_\alpha} + i\bar\theta_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu\right]$$

$$\bar Q_{\dot\alpha} = -i\left[\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\right]$$

The SUSY algebra is fulfilled

# R-symmetry

The SUSY algebra is invariant under:

$$Q_\alpha \to Q'_\alpha = e^{i\beta} Q_\alpha \qquad \bar{Q}_{\dot\alpha} \to \bar{Q}'_{\dot\alpha} = e^{-i\beta} \bar{Q}_{\dot\alpha}$$

Generated by R ⟹ It does not commute with SUSY

$$[Q_\alpha , R] = Q_\alpha \qquad [\bar{Q}_{\dot\alpha} , R] = -\bar{Q}_{\dot\alpha}$$

$Q_\alpha$ has $R$-charge $+1$ $\qquad$ $\bar{Q}_{\dot\alpha}$ has R-charge -1

Since:

$$Q \sim \frac{\partial}{\partial\theta} + \bar\theta\,\partial_\mu \qquad \bar{Q} \sim \frac{\partial}{\partial\bar\theta} + \theta\,\partial_\mu$$

The R-symmetry is realized in superspace as

$$\theta \to e^{-i\beta}\theta \,, \qquad \bar\theta \to e^{i\beta}\bar\theta$$