Supersymmetry

Lecture 3

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

Alfonso V. Ramallo

Frequentely used identities in superspace

$$\theta^\alpha \theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\,\theta\theta \qquad \theta_\alpha \theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\,\theta\theta$$

$$\bar\theta^{\dot\alpha} \bar\theta^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\,\bar\theta\bar\theta \qquad \bar\theta_{\dot\alpha} \bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\,\bar\theta\bar\theta$$

$$\theta\psi\;\theta\chi = -\frac{1}{2}\theta\theta\;\psi\chi \qquad \bar\theta\bar\psi\;\bar\theta\bar\chi = -\frac{1}{2}\bar\theta\bar\theta\;\bar\psi\bar\chi$$

From the completeness property of the Pauli matrices

$$\left(\sigma^{\mu}\right)_{\alpha\dot{\alpha}}\left(\bar{\sigma}_{\mu}\right)^{\dot{\beta}\beta} = 2\,\delta_{\alpha}^{\beta}\,\delta_{\dot{\alpha}}^{\dot{\beta}}$$

We get

$$\theta^{\alpha}\,\bar{\theta}^{\dot{\alpha}} = \frac{1}{2}\left(\bar{\sigma}^{\mu}\right)^{\dot{\alpha}\alpha}\,\theta\sigma_{\mu}\,\bar{\theta}$$

Other properties:

$$\theta\sigma^{\mu}\,\bar{\theta}\;\theta\sigma^{\nu}\,\bar{\theta} = \frac{1}{2}\,\theta\theta\;\bar{\theta}\bar{\theta}\,g^{\mu\nu}$$

$$\theta\sigma^{\mu}\bar{\eta} = -\bar{\eta}\bar{\sigma}^{\mu}\,\theta$$

Function in superspace = superfield

A superfield in components

$$\mathcal{F}(x,\theta,\bar\theta) = f(x) + (\theta\varphi(x)) + (\bar\theta\bar\chi(x)) + (\theta\theta)m(x) + (\bar\theta\bar\theta)n(x) +$$
$$+\theta\sigma^\mu\bar\theta v_\mu(x) + (\theta\theta)(\bar\theta\bar\lambda(x)) + (\bar\theta\bar\theta)(\theta\psi(x)) + (\theta\theta)(\bar\theta\bar\theta)d(x)$$

$f, m, n, d \longrightarrow$ complex scalar fields

$v_\mu \longrightarrow$ complex vector field

$\varphi_\alpha, \bar\chi^{\dot\alpha}, \bar\lambda^{\dot\alpha}, \psi_\alpha \longrightarrow$ spinor fields

Too many!!!

These fields represent the SUSY algebra

$$\delta_S\,\mathcal{F}(x,\theta,\bar\theta) = \left[-i\eta^\alpha Q_\alpha - i\bar\eta_{\dot\alpha}\bar Q^{\dot\alpha}\right]\mathcal{F}(x,\theta,\bar\theta) = \delta_S f + \theta\,\delta_S\varphi + \bar\theta\delta_S\bar\chi + \cdots$$

To get a shorter representation of SUSYwe impose a constraint:

$$\mathcal{O}\,\mathcal{F} = 0$$ differential operator in superspace

Compatibility with SUSY

$$\mathcal{O}\,\mathcal{F} = 0 \quad \Longrightarrow \quad \mathcal{O}\left[\delta_S\,\mathcal{F}\right] = 0$$

Guaranteed if

$$\mathcal{O}\left[\delta_S\,\mathcal{F}\right] = \delta_S\left[\mathcal{O}\,\mathcal{F}\right]$$

Define the superspace covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$$

$$\bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

They anticommute with SUSY generators

$$\{D_\alpha\,,\,Q_\beta\} = \{D_\alpha\,,\,\bar Q_{\dot\beta}\} = \{\bar D_{\dot\alpha}\,,\,Q_\beta\} = \{\bar D_{\dot\alpha}\,,\,\bar Q_{\dot\beta}\} = 0$$

They satisfy the SUSY algebra with the wrong sign

$$\{ D_\alpha \, , \bar{D}_{\dot{\alpha}} \} \; = \; -2i \, \sigma^{\mu}_{\alpha\dot{\alpha}} \, \partial_\mu \qquad\qquad \{ D_\alpha \, , \, D_\beta \} \; = \; \{ \bar{D}_{\dot{\alpha}} \, , \, \bar{D}_{\dot{\beta}} \} \; = \; 0$$

Chiral superfield

$$\bar{D}_{\dot{\alpha}} \Phi = 0$$

Notice the similarity with analytic functions!

Solving the constraint

$$\bar{D}_{\dot{\alpha}} \theta^\alpha = 0$$

$$y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta} \quad \Longrightarrow \quad \bar{D}_{\dot{\alpha}} \, y^\mu = 0$$

Solution:

$$\Phi = A(y) + \sqrt{2}(\theta\psi(y)) + (\theta\theta)F(y)$$

Expansion in components:

$$\Phi = A(y) + \sqrt{2}(\theta\psi(y)) + (\theta\theta)F(y)$$

Physical content:

$$A \implies \text{complex scalar field}$$

$$\psi_\alpha \implies \text{Weyl spinor field}$$

$$F \implies \text{auxiliary scalar field}$$

In terms of the original variables

$$\Phi = A(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu A(x) +$$

$$+\sqrt{2}(\theta\psi(x)) + \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi(x)\sigma^\mu\bar{\theta}) + (\theta\theta)F(x)$$

SUSY generators acting on functions of y

$$\frac{\partial}{\partial \theta^\alpha}\,\Phi(x,\theta,\bar\theta) = \left[\frac{\partial}{\partial \theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\partial_\mu\right]\Phi(y,\theta)$$

$$\frac{\partial}{\partial \bar\theta_{\dot\alpha}}\,\Phi(x,\theta,\bar\theta) = \left[\frac{\partial}{\partial \bar\theta_{\dot\alpha}} + i(\bar\sigma^\mu)^{\dot\alpha\alpha}\theta_\alpha\partial_\mu\right]\Phi(y,\theta)$$

$$\boxed{Q_\alpha = i\,\frac{\partial}{\partial \theta^\alpha}}$$

$$\boxed{\bar Q^{\dot\alpha} = i\left[\frac{\partial}{\partial \bar\theta_{\dot\alpha}} + 2i(\bar\sigma^\mu)^{\dot\alpha\alpha}\theta_\alpha\partial_\mu\right]}$$

SUSY tranformation in components:

$$\delta_S A = \sqrt{2}\,\eta\,\psi$$

$$\delta_S \psi_\alpha = \sqrt{2}\,\eta_\alpha\, F - i\sqrt{2}\,\sigma^\mu_{\alpha\dot{\alpha}}\,\bar{\eta}^{\dot{\alpha}}\,\partial_\mu A$$

$$\delta_S F = i\sqrt{2}\,\partial_\mu \psi\,\sigma^\mu\,\bar{\eta} = -i\sqrt{2}\,\bar{\eta}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

$\delta_S F$ is a total derivative

Berezin integration

Let $f(\theta)$ be:

$$f(\theta) = a_0 + \theta a_1$$

$$\int \frac{df}{d\theta} d\theta = 0 \quad \Rightarrow \quad \boxed{\int d\theta = 0} \quad \boxed{\int \theta d\theta = 1}$$

$$\boxed{\int f(\theta) \, d\theta = \frac{df}{d\theta} = a_1}$$

For two Grassmann variables θ^α with $\alpha = 1, 2$, we define:

$$\int d^2\theta \equiv \frac{1}{2} \int d\theta^2 \, d\theta^1$$

$$\int d^2\theta \; 1 = 0 \qquad\qquad \int d^2\theta \; \theta^\alpha = 0$$

The only non-zero integral in θ is:

$$\boxed{\int d^2\theta \; \theta\theta = \frac{1}{2} \int d\theta^2 \, d\theta^1 \, (2\theta^1\theta^2) = 1}$$

Similarly for $\bar{\theta}$ the only non-zero integral is:

$$\boxed{\int d^2\bar{\theta} \; \bar{\theta}\bar{\theta} = 1}$$

Let us consider the integral of a superfield

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \mathcal{F}(x, \theta, \bar{\theta})$$

SUSY variation:

$$\delta_S\, S = -i \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \left[\eta\, Q + \bar{\eta}\, \bar{Q}\right] \mathcal{F}$$

Since Q and $\bar{Q}$ involve derivatives $\frac{\partial}{\partial x^\mu}$, $\frac{\partial}{\partial \theta^\alpha}$ and $\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}$, one has automatically:

$$\boxed{\delta_S\, S = 0}$$

SUSY invariant

Notice

$$S = \int d^4x\, \mathcal{F}_{|_{\theta^2\bar{\theta}^2}} = \frac{1}{16} \int d^4x\, D^2\, \bar{D}^2 \mathcal{F}_{|_0}$$

Consider

$$\mathcal{F} = \Phi^\dagger \, \Phi$$

Since:

$$(\chi\sigma^\mu \bar{\psi})^\dagger = \psi\sigma^\mu \bar{\chi} \qquad (\theta\sigma^\mu\bar{\theta})^\dagger = \theta\sigma^\mu\bar{\theta} \qquad (\partial_\mu \psi \, \sigma^\mu \, \bar{\theta})^\dagger = \theta\sigma^\mu\partial_\mu\bar{\psi}$$

$$\Phi^\dagger = \bar{A}(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\bar{A}(x) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\bar{A}(x) + \\ + \sqrt{2}(\bar{\theta}\bar{\psi}(x)) - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})(\theta\sigma^\mu\partial_\mu\bar{\psi}(x)) + (\bar{\theta}\bar{\theta})\bar{F}(x)$$

$\Phi^\dagger$ is an antichiral superfield

$$D^\alpha \, \Phi^\dagger = 0$$

$$\Phi^\dagger \Phi|_{\theta^2\bar\theta^2} = \bar F F - \frac{1}{4}\bar A \partial_\mu \partial^\mu A - \frac{1}{4}\partial_\mu\partial^\mu \bar A A +$$

$$+\frac{1}{2}\partial_\mu \bar A \partial^\mu A + \frac{i}{2}\psi\sigma^\mu \partial_\mu \bar\psi - \frac{i}{2}\partial_\mu \psi \sigma^\mu \bar\psi$$

$$\Downarrow$$

$$\Phi^\dagger \Phi|_{\theta^2\bar\theta^2} = \partial_\mu \bar A \partial^\mu A + i\bar\psi \bar\sigma^\mu \partial_\mu \psi + \bar F F + \text{total derivatives}$$

Action:

$$S = \int d^4x \left[\partial_\mu \bar A \partial^\mu A + i\bar\psi\bar\sigma^\mu \partial_\mu \psi + \bar F F \right]$$

F is an auxiliary field

S is invariant under SUSY

Adding interactions

The action contains a D-term and a F-term

$$S = \int d^4x \int d^2\theta \, d^2\bar{\theta} \;\; \Phi^\dagger(x,\theta,\bar{\theta}) \, \Phi(x,\theta,\bar{\theta}) + \left[\int d^4y \int d^2\theta \, W[\Phi(y,\theta)] + h.c. \right]$$

$W[\Phi]$ is called the superpotential

$W[\Phi]$ is chiral if Φ is chiral ➡ Action automatically SUSY invariant

D-term $\Longrightarrow \int d^2\theta \, d^2\bar{\theta} \, [\cdots]$ ➡ kinetic terms

F-term $\Longrightarrow \int d^2\theta \, [\cdots] \, + \, h.c$ ➡ mass terms and interactions

By Taylor expansion

$$W[\Phi] = W[A] + \sqrt{2}\,\frac{\partial W}{\partial A}\,\theta\psi + \theta\theta\left[F\,\frac{\partial W}{\partial A} - \frac{1}{2}\,\frac{\partial^2 W}{\partial A^2}\,(\psi\psi)\right]$$

Then:

$$\left[\int d^4y \int d^2\theta\, W[\Phi(y,\theta)] + h.c.\right] = \int d^4y \left[F\,\frac{\partial W}{\partial A} - \frac{1}{2}\,\frac{\partial^2 W}{\partial A^2}\,(\psi\psi) + h.c.\right]$$

$$S = \int d^4x \left[\partial_\mu \bar{A}\,\partial^\mu A + i\,\bar{\psi}\bar{\sigma}^\mu\,\partial_\mu \psi + \bar{F}\,F + \left[F\,\frac{\partial W}{\partial A} - \frac{1}{2}\,\frac{\partial^2 W}{\partial A^2}\,(\psi\psi) + h.c.\right]\right]$$

Invariant under SUSY for any W !!

The equation of motion of F is:

$$\bar{F} = -\frac{\partial W}{\partial A} \quad \Rightarrow \quad F = -\overline{\frac{\partial W}{\partial A}}$$

In the action

$$\bar{F} F + F \frac{\partial W}{\partial A} + \bar{F} \overline{\frac{\partial W}{\partial A}} = -\left| \frac{\partial W}{\partial A} \right|^2$$

Lagrangian density

$$\mathcal{L} = \partial_\mu \bar{A} \partial^\mu A + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \left| \frac{\partial W}{\partial A} \right|^2 - \frac{1}{2} \frac{\partial^2 W}{\partial A^2} (\psi\psi) - \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar{A}^2} (\bar{\psi}\bar{\psi})$$

potential $\Rightarrow$

$$V = \left| \frac{\partial W}{\partial A} \right|^2 = |F|^2$$

For renormalizable interactions W should be at most cubic

SUSY vacuum

$$V = 0 \rightarrow F = 0 \rightarrow \frac{\partial W}{\partial A} = 0$$

F-term condition

If there is no solution SUSY is broken

Example:

$$W[\Phi] = a\Phi \rightarrow \frac{\partial W}{\partial A} = a \neq 0$$

This is the O'Raifeartaigh mechanism (F-term breaking)

Wess-Zumino model

superpotential $\Rightarrow$ $$W[\Phi] = \frac{m}{2}\,\Phi^2 + \frac{g}{3}\,\Phi^3$$

potential $\Rightarrow$ $$V = m^2\,|A|^2 + gm\,[\bar{A}A^2 + A\bar{A}^2] + g^2\,|A|^4$$

Lagrangian

$$\mathcal{L} = \partial_\mu \bar{A}\,\partial^\mu A + i\bar{\psi}\bar{\sigma}^\mu\,\partial_\mu\psi - \frac{m}{2}\,(\psi\psi + \bar{\psi}\bar{\psi}) - m^2\,|A|^2 - \\ -gm\,[\bar{A}A^2 + A\bar{A}^2] - g^2\,|A|^4 - g\,(A\,\psi\psi + \bar{A}\,\bar{\psi}\bar{\psi})$$

Relation between quartic and Yukawa couplings as the one needed to solve the hierarchy problem !!