Supersymmetry

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Lecture 5

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

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We want to construct the action for the gauge field and its SUSY partner

We need the SUSY analogue to the field strength

Define:

$$W_\alpha = -\frac{1}{8}\,\bar{D}\bar{D}e^{-2V}\,D_\alpha e^{2V}$$

$$\bar{W}_{\dot\alpha} = \frac{1}{8}\,DDe^{2V}\,\bar{D}_{\dot\alpha}e^{-2V}$$

They satisfy:

$$W_\alpha^\dagger = \bar{W}_{\bar\alpha}$$

W_α is chiral because $\bar{D}^3 = 0$:

$$\boxed{\bar{D}_{\dot{\alpha}} W_\beta = 0}$$

Similarly $\bar{W}_{\dot{\alpha}}$ is antichiral:

$$\boxed{D_\alpha \bar{W}_{\dot{\beta}} = 0}$$

W_α transforms covariantly under gauge transformations

$$e^{2V} \rightarrow e^{2V'} = e^{-2i\Lambda^\dagger} e^{2V} e^{2i\Lambda}$$
$$e^{-2V} \rightarrow e^{-2V'} = e^{-2i\Lambda} e^{2V} e^{2i\Lambda^\dagger}$$

$\Rightarrow$

$$\boxed{W_\alpha \rightarrow W'_\alpha = e^{-2i\Lambda} W_\alpha e^{2i\Lambda}}$$

W_α remains chiral

Similarly

$$\boxed{\bar{W}_{\bar{\alpha}} \rightarrow \bar{W}'_{\dot{\alpha}} = e^{-2i\Lambda^\dagger} \bar{W}_{\dot{\alpha}} e^{2i\Lambda^\dagger}}$$

Gauge invariant traces:

$$\mathrm{Tr}\left[W^\alpha W_\alpha \right] \qquad \mathrm{Tr}\left[\bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right]$$

Candidates for the gauge field action

Abelian case

$$W_\alpha = -\frac{1}{4}\,\bar{D}\,\bar{D}\,D_\alpha\,V$$

$$\bar{W}_{\dot\alpha} = -\frac{1}{4}\,D\,D\,\bar{D}_{\dot\alpha}\,V$$

$$\bar{D}_{\dot\alpha}\,\bar{W}^{\dot\alpha} = D^\alpha\,W_\alpha$$

Non-abelian case (WZ gauge)

$$W_\alpha = -\frac{1}{4}\,\bar{D}\,\bar{D}\Big[D_\alpha\,V + [D_\alpha\,V, V]\Big]$$

$$\bar{W}_{\dot\alpha} = -\frac{1}{4}\,D\,D\Big[\bar{D}_{\dot\alpha}\,V - [\bar{D}_{\dot\alpha}\,V, V]\Big]$$

$$D_\alpha V = \left(\sigma^\mu \bar\theta\right)_\alpha A_\mu + 2i\,\theta_\alpha\,(\bar\theta\bar\lambda) - i\bar\theta^2\,\lambda_\alpha + \theta_\alpha\,\bar\theta^2\,D +$$

$$-i\bar\theta^2\left[\sigma^{\mu\nu}\,\theta\right]_\alpha\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right] - \theta^2\,\bar\theta^2\left(\sigma^\mu\,\partial_\mu\,\bar\lambda\right)_\alpha$$

$$[D_\alpha V, V] = \bar\theta^2\left(\sigma^{\mu\nu}\,\theta\right)_\alpha\left[A_\mu\,,\,A_\nu\right] - i\theta^2\bar\theta^2\left[A_\mu\,,\,(\sigma^\mu\bar\lambda)_\alpha\right]$$

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha\,D - i\left(\sigma^{\mu\nu}\,\theta\right)_\alpha F_{\mu\nu} - \theta^2\left(\sigma^\mu\,D_\mu\,\bar\lambda\right)_\alpha$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu\,,\,A_\nu\right]$$

$$D_\mu\,\bar\lambda = \partial_\mu\,\bar\lambda + i\left[A_\mu\,,\,\bar\lambda\right]$$

$$\bar{W}^{\dot{\alpha}} = i\,\bar{\lambda}^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\,D + i\left(\bar{\sigma}^{\mu\nu}\,\bar{\theta}\right)^{\dot{\alpha}} F_{\mu\nu} + \bar{\theta}^2\,\left(\bar{\sigma}^{\mu}\,D_\mu\,\lambda\right)^{\dot{\alpha}}$$

gauge invariant traces

$$\mathrm{Tr}\left[W^\alpha\,W_\alpha\right]\Big|_{\theta^2} = \theta^2\;\mathrm{Tr}\left[D^2 + 2i\,\bar{\lambda}\,\bar{\sigma}^\mu\,D_\mu\,\lambda - \frac{1}{2}\left(F_{\mu\nu}\,F^{\mu\nu} - i\,F_{\mu\nu}\,{}^*F^{\mu\nu}\right)\right]$$

$$\mathrm{Tr}\left[\bar{W}_\alpha\,\bar{W}^\alpha\right]\Big|_{\bar{\theta}^2} = \bar{\theta}^2\;\mathrm{Tr}\left[D^2 + 2i\,\bar{\lambda}\,\bar{\sigma}^\mu\,D_\mu\,\lambda - \frac{1}{2}\left(F_{\mu\nu}\,F^{\mu\nu} + i\,F_{\mu\nu}\,{}^*F^{\mu\nu}\right)\right]$$

$${}^*F^{\mu\nu} \equiv \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\;F_{\rho\sigma}$$

Introduction of coupling constant:

$$\frac{1}{4g^2}\left[\int d^2\theta\,\mathrm{Tr}\left[W^\alpha\,W_\alpha\right]+\int d^2\bar{\theta}\,\mathrm{Tr}\left[\bar{W}_{\dot{\alpha}}\,\bar{W}^{\dot{\alpha}}\right]\right]=$$

$$-\frac{1}{4g^2}\,\mathrm{Tr}\left[F_{\mu\nu}\,F^{\mu\nu}\right]+\frac{i}{g^2}\,\mathrm{Tr}\left[\bar{\lambda}\,\bar{\sigma}^\mu\,D_\mu\,\lambda\right]+\frac{1}{2g^2}\,\mathrm{Tr}\left[D^2\right]$$

Define the complexified coupling as:

$$\tau=\frac{\theta_{YM}}{2\pi}+\frac{4\pi i}{g^2}$$

$$\mathcal{L}_{gauge}=-\frac{i}{16\pi}\int d^2\theta\,\tau\,\mathrm{Tr}\left[W^\alpha\,W_\alpha\right]+\frac{i}{16\pi}\int d^2\bar{\theta}\,\bar{\tau}\,\mathrm{Tr}\left[\bar{W}_{\dot{\alpha}}\,\bar{W}^{\dot{\alpha}}\right]$$

Equivalently:

$$\mathcal{L}_{gauge} = \frac{1}{8\pi}\,\mathrm{Im}\left[\tau\int d^2\theta\,\mathrm{Tr}\left[W^\alpha\,W_\alpha\right]\right]$$

In components:

$$\mathcal{L}_{gauge} = -\frac{1}{4g^2}\,\mathrm{Tr}\left[F_{\mu\nu}\,F^{\mu\nu}\right] + \frac{\theta_{YM}}{32\pi^2}\,\mathrm{Tr}\left[F_{\mu\nu}\,{}^*F^{\mu\nu}\right] +$$

$$+\frac{i}{g^2}\,\mathrm{Tr}\left[\bar\lambda\,\bar\sigma^\mu\,D_\mu\,\lambda\right] + \frac{1}{2g^2}\,\mathrm{Tr}\left[D^2\right]$$

Coupled matter+gauge fields

Chiral matter superfields $\Phi_i = (\varphi_i, \psi_i, F_i)$

$\varphi_i \longrightarrow$ complex scalars $\quad \psi_i \longrightarrow$ chiral fermions

$F_i \longrightarrow$ auxiliary fields $\quad i \longrightarrow$ flavor index

$$\mathcal{L}_{matter} = \int d^2\theta \, \bar{d}^2\theta \sum_i \Phi_i^\dagger \, e^{2V} \, \Phi_i + \int d^2 \, \theta \, W[\Phi_i] + \int d^2\bar{\theta} \, W^\dagger[\Phi_i]$$

$$\begin{aligned}\mathcal{L}_{matter} = & \left(D_\mu \varphi_i \right)^\dagger \left(D^\mu \varphi_i \right) + i\bar{\psi}_i \, \bar{\sigma}^\mu \, D_\mu \, \psi_i + F_i^\dagger \, F_i + \\ & + \varphi_i^\dagger \, D \, \varphi_i + i\sqrt{2} \, \varphi_i^\dagger \lambda \psi_i - i\sqrt{2} \, \bar{\psi}_i \, \bar{\lambda} \, \varphi_i + \\ & + F_i \frac{\partial W}{\partial \varphi_i} - \frac{1}{2} \frac{\partial^2 W}{\partial \varphi_i \, \partial \varphi_j} \psi_i \, \psi_j + \text{h.c.}\end{aligned}$$

Fayet-Iliopoulos term

$V^A \rightarrow$ vector superfield along an abelian direction

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\, V^A \Longrightarrow$$ SUSY invariant

Also gauge invariant $\quad V^A \rightarrow V^A + i(\Lambda - \Lambda^\dagger)$

$$\Lambda_{|_{\theta^2\bar{\theta}^2}} \,,\, \Lambda^\dagger_{|_{\theta^2\bar{\theta}^2}} = \partial_\mu \partial^\mu \,(\text{something})$$

$$\mathcal{L}_{FI} = 2 \sum_{A \in Abelian} \xi^A \int d^2\theta\, d^2\bar{\theta}\, V^A$$

In the WZ gauge

$$\mathcal{L}_{FI} = \sum_A \xi^A D^A$$

Total lagrangian

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{FI}$$

equation of motion of F

$$F_i^\dagger = -\frac{\partial W}{\partial \varphi_i}$$

Terms with D in $\mathcal{L}$:

$$\frac{1}{2g^2} D^a D^a + D^a \sum_i \varphi_i^\dagger T^a \varphi_i + \xi^a D^a$$

($\xi^a = 0$ if a is not in an abelian factor)

$$D^a = -g^2 \left[\sum_i \varphi_i^\dagger T^a \varphi_i + \xi^a \right]$$

on-shell contribution

$$\left[\frac{1}{2g^2}\, D^a\, D^a \;+\; D^a \sum_i \varphi_i^\dagger\, T^a\, \varphi_i \;+\; \xi^a\, D^a\right]_{on-shell} \;=\; -\frac{1}{2g^2}\, D^a\, D^a$$

Potential

$$V \;=\; \sum_i F_i^\dagger\, F_i \;+\; \frac{1}{2g^2}\, D^a\, D^a$$

$$\Downarrow$$

$$V(\varphi_i\,,\,\varphi_i^\dagger) \;=\; \sum_i \left|\frac{\partial W}{\partial \varphi_i}\right|^2 \;+\; \frac{g^2}{2} \sum_a \left[\,\sum_i \varphi_i^\dagger\, T^a\, \varphi_i \;+\; \xi^a\,\right]^2$$

Total lagrangian without auxiliary fields

$$\mathcal{L} = -\frac{1}{4g^2}\,\mathrm{Tr}\left[\,F_{\mu\nu}\,F^{\mu\nu}\,\right] + \frac{\theta_{YM}}{32\pi^2}\,\mathrm{Tr}\left[\,F_{\mu\nu}\,{}^*F^{\mu\nu}\,\right] + \frac{i}{g^2}\,\mathrm{Tr}\left[\,\bar{\lambda}\,\bar{\sigma}^\mu\,D_\mu\,\lambda\,\right] +$$

$$+\left(\,D_\mu\varphi_i\,\right)^\dagger\left(\,D^\mu\varphi_i\,\right) + i\bar{\psi}_i\,\bar{\sigma}^\mu\,D_\mu\,\psi_i + i\sqrt{2}\,\varphi_i^\dagger\lambda\psi_i - i\sqrt{2}\,\bar{\psi}_i\,\bar{\lambda}\,\varphi_i +$$

$$-\frac{1}{2}\,\frac{\partial^2 W}{\partial\varphi_i\,\partial\varphi_j}\,\psi_i\,\psi_j - \frac{1}{2}\,\frac{\partial^2 \bar{W}}{\partial\varphi_i^\dagger\,\partial\varphi_j^\dagger}\,\bar{\psi}_i\,\bar{\psi}_j - V(\varphi_i\,,\,\varphi_i^\dagger)$$

SUSY vacua

$$\frac{\partial W}{\partial \varphi_i} = 0$$

F-flatness

$$\sum_i \varphi_i^\dagger T^a \varphi_i + \xi^a = 0$$

D-flatness

- If there is no solution SUSY is spontaneously broken
- If there is a solution with $< \varphi_i > \neq 0$ the gauge symmetry is spontaneously broken and the gauge field acquires mass
- If there is a continous set of vacuum solutions they form a manifold (moduli space)

SQCD

We must get non-chiral matter content from chiral fields

In terms of Dirac spinors

$$\bar{\Psi} = \left(\chi^{\alpha} \, , \, \bar{\psi}_{\dot{\alpha}} \right) \qquad \Psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

kinetic energy

$$i\bar{\Psi} \, \gamma^{\mu} \, \partial_{\mu} \, \Psi = i \left[\bar{\psi} \bar{\sigma}^{\mu} \, \partial_{\mu} \, \psi + \bar{\chi} \, \bar{\sigma}^{\mu} \, \partial_{\mu} \, \chi \right]$$

Interaction term

$$A^a_{\mu} \, \bar{\Psi} \, \gamma^{\mu} \, T^a \, \Psi = A^a_{\mu} \left[\bar{\psi} \bar{\sigma}^{\mu} \, T^a \, \psi - \bar{\chi} \, \bar{\sigma}^{\mu} \, T^a \, \chi \right]$$

different charge sign!

quark chiral superfied

$$Q = \varphi(y) + \sqrt{2} \, \theta \, \psi(y) + \theta^2 \, F(y)$$

antiquark chiral superfied

$$\tilde{Q} = \tilde{\varphi}(y) + \sqrt{2} \, \theta \, \tilde{\psi}(y) + \theta^2 \, \tilde{F}(y) \qquad \tilde{\psi} \equiv \chi$$

Matter lagrangian

$$\mathcal{L}_{matter} = \int d^2\theta \, d^2\bar{\theta} \left[Q^\dagger \, e^{2V} \, Q + \tilde{Q} \, e^{-2V} \, \tilde{Q}^\dagger \right]$$

Index structure

$$Q \to Q^i \qquad \tilde{Q} \to \tilde{Q}_i \qquad Q^\dagger \to Q^\dagger_i \qquad \tilde{Q}^\dagger \to \tilde{Q}^{\dagger\, i} \qquad V \to V^i{}_j$$

$$Q \to \text{fundamental rep} \qquad \tilde{Q} \to \text{antifundamental rep}$$

Gauge transformations

$$Q \to e^{-2i\Lambda} \, Q \qquad \tilde{Q} \to \tilde{Q} \, e^{2i\Lambda} \qquad Q^\dagger \to Q^\dagger \, e^{2i\Lambda^\dagger} \qquad \tilde{Q}^\dagger \to e^{-2i\Lambda^\dagger} \, \tilde{Q}^\dagger$$

$$e^{2V} \to e^{-2i\Lambda^\dagger} \, e^{2V} \, e^{2i\Lambda} \qquad e^{-2V} \to e^{-2i\Lambda} \, e^{-2V} \, e^{2i\Lambda^\dagger}$$

Global rotation with $\Lambda = \alpha$ with $\alpha = \alpha^\dagger$ and α constant

In terms of the Dirac spinor

$$\left.\begin{array}{l} \psi \to e^{-2i\alpha}\,\psi \\ \chi \to \chi\, e^{2i\alpha} \Longrightarrow \bar{\chi} \to e^{-2i\alpha}\,\bar{\chi} \end{array}\right\} \Longrightarrow \Psi \to e^{-2i\alpha}\Psi$$

Superpotential mass term

$$\boxed{W(Q,\tilde{Q}) = m\,\tilde{Q}\,Q}$$

$\Longrightarrow$ gauge invariant

Contribution to the lagrangian

$$m[\,\psi\tilde{\psi} + \bar{\psi}\bar{\tilde{\psi}}\,] = m\bar{\Psi}\,\Psi$$

OK!

Terms with D in the lagrangian

$$\frac{1}{2g^2}\, D^a\, D^a \;+\; D^a \left[\varphi^\dagger\, T^a\, \varphi \;-\; \tilde{\varphi}\, T^a\, \tilde{\varphi}^\dagger \right] + \xi^a\, D^a$$

Then:

$$D^a \;=\; -g^2 \left[\varphi^\dagger\, T^a\, \varphi \;-\; \tilde{\varphi}\, T^a\, \tilde{\varphi}^\dagger \;+\; \xi^a \right]$$

Potential

$$V \;=\; \left| \frac{\partial W}{\partial \varphi} \right|^2 + \left| \frac{\partial W}{\partial \tilde{\varphi}} \right|^2 + \frac{g^2}{2} \sum_a \left[\varphi^\dagger\, T^a\, \varphi \;-\; \tilde{\varphi}\, T^a\, \tilde{\varphi}^\dagger \;+\; \xi^a \right]^2$$

SUSY QED

Let us consider the $U(1)$ case with mass m

$$V = m^2 \left[|\varphi|^2 + |\tilde{\varphi}|^2 \right] + \frac{1}{2} \left[|\varphi|^2 - |\tilde{\varphi}|^2 + \xi \right]^2$$

$$V = \xi^2 + (m^2 + \xi)|\varphi|^2 + (m^2 - \xi)|\tilde{\varphi}|^2 + \frac{1}{2} \left[|\varphi|^2 - |\tilde{\varphi}|^2 \right]^2$$

$m, \xi \neq 0 \implies$ SUSY broken

$m^2 \geq \xi \implies U(1)$ unbroken $(<\varphi>=<\tilde{\varphi}>=0)$

$m^2 < \xi \implies U(1)$ broken $(<\varphi>=0\ ,\ <\tilde{\varphi}>\neq 0)$

For $m^2, \xi = 0$, the SUSY vacuum equations are

$$|\varphi|^2 - |\tilde{\varphi}|^2 = 0 \quad \Longrightarrow \quad \varphi = e^{i\alpha}\,\tilde{\varphi}\ ,\ \ \alpha \in \mathbb{R}$$

$U(1)$ symmetry $\varphi \to e^{i\beta}\varphi$, $\tilde{\varphi} \to e^{-i\beta}\tilde{\varphi}$

Take $\beta = \alpha/2 \ \to \varphi = \tilde{\varphi}$ $\Longleftarrow$ $\beta = \pi + \frac{\alpha}{2}$ is also good

Moduli space

$$\mathcal{M} = \{\varphi/\varphi \to -\varphi\}$$

Define the "meson" field

$$M = \varphi\tilde{\varphi}$$

It is a good coordinate for $\mathcal{M}$

$$\mathcal{M} = \{M\}$$

Metric of $\mathcal{M}$

Kahler potential $$K = \varphi^* \varphi + \tilde{\varphi}^* \tilde{\varphi} = 2\varphi^* \varphi = 2\sqrt{M M^*}$$

$$ds^2 = \frac{\partial^2 K}{\partial M \, \partial M^*} \, dM \, dM^* = \frac{1}{2} \, \frac{dM \, dM^*}{\sqrt{M \, M^*}}$$

Parametrize M as:

$$M = r \, e^{i\theta} \quad \Longrightarrow \quad ds^2 = \frac{1}{2} \left[\frac{dr^2}{r} + r \, d\theta^2 \right]$$

change to new variables $$u = \sqrt{2r} \, , \qquad \phi = \theta/2$$

$$\boxed{ds^2 = du^2 + u^2 \, d\phi^2 \, , \qquad 0 \leq u < \infty \, , \; 0 \leq \phi < \pi}$$

$\mathbb{C}/\mathbb{Z}_2$ with a conical singularity at the origin

The singularity occurs where the foton is massless