Lecture 6: Extended supersymmetry

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SUPERSYMMETRY

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Extended supersymmetry

Theoretical and mathematical reasons to study (extended) supersymmetry:

- Exact results for QFT due to duality properties and holomorphicity.
- Essential ingredient in superstring theory.
- SUSY theories can be converted into topological field theories and used to classify manifolds in topology and algebraic geometry.
- $\cdots$

From now on we will follow this track. We leave a detailed discussion of the (phenomenologically relevant) minimal SUSY model for Javier's part.

It is possible to have more than one supercharge, say, $\mathcal{N}$,

$$\mathcal{Q}^I_\alpha \qquad \text{and} \qquad \bar{\mathcal{Q}}^I_{\dot\alpha} = \left(\mathcal{Q}^I_\alpha\right)^\dagger \qquad I = 1, \ldots, \mathcal{N}$$

Notice that the number of real supercharge components is $4\mathcal{N}$.

Sometimes it is useful to say that the theory has $4\mathcal{N}$ supersymmetries or supercharges; alternatively, it is $\mathcal{N}$ extended supersymmetric.

Extended (super) Poincaré algebra

The anticommutator $\{\mathcal{Q}, \bar{\mathcal{Q}}\}$ transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation, thus it has to be proportional to P_μ

$$\left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^J_{\dot\beta}\right\} = 2\, C^{IJ}\, (\sigma^\mu)_{\alpha\dot\beta}\, P_\mu$$

But σ^μ are Hermitian, as well as the supercharges, then $C^{IJ} = C^{JI\star}$, *i.e.*, C is Hermitian. Then, there is U,

$$\mathcal{Q}^I_\alpha \to {U^I}_K\, \mathcal{Q}^K_\alpha \qquad \bar{\mathcal{Q}}^J_{\dot\beta} \to \bar{\mathcal{Q}}^L_{\dot\beta}\, {(U^{-1})_L}^J$$

that diagonalizes C = diag(c_I). Now, $c_I > 0$ (positivity of the energy), thus

$$\mathcal{Q}^I_\alpha \to \sqrt{c_I}\, \mathcal{Q}^I_\alpha \qquad \bar{\mathcal{Q}}^J_{\dot\beta} \to \sqrt{c_J}\, \bar{\mathcal{Q}}^J_{\dot\beta}$$

and we get the bracket

$$\left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^J_{\dot\beta}\right\} = 2\, \delta^{IJ}\, (\sigma^\mu)_{\alpha\dot\beta}\, P_\mu$$

like $\mathcal{N}$ copies of the minimal supersymmetry.

Extended (super) Poincaré algebra

$\{\mathcal{Q}, \mathcal{Q}\}$ must be a linear combination of bosonic operators in the $(0,0)$ and $(1,0)$ representations of the Lorentz group.

The only $(1,0)$ is the self-dual part of $M_{\mu\nu}$, but it would not commute with P_μ.

Thus, we need a new *generator*, $\mathcal{Z}_{IJ}$

$$\left\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\right\} = 2\,\epsilon_{\alpha\beta}\;\mathcal{Z}^{IJ} \qquad\qquad \mathcal{Z}_{IJ} = -\mathcal{Z}_{JI}$$

that should be a linear combination of the internal symmetry generators,

$$\mathcal{Z}_{IJ} = (a^a_{IJ})\;T^a$$

$\mathcal{Z}^{IJ}$ are central extensions or central charges (which can be deduced from the algebra and the Jacobi identities) $\quad\Rightarrow\quad \mathcal{Z}^{IJ} \in \mathrm{Z}(\mathcal{G})$.

The adjoint of the bracket above reads

$$\left\{\bar{\mathcal{Q}}^I_{\dot\alpha}, \bar{\mathcal{Q}}^J_{\dot\beta}\right\} = -2\,\epsilon_{\dot\alpha\dot\beta}\;\mathcal{Z}^{IJ\,\dagger}$$

where we used $\epsilon_{\dot\alpha\dot\beta} = -\epsilon_{\alpha\beta}$.

Extended (super) Poincaré algebra

$$[P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, M_{\rho\sigma}] = i\,(\eta_{\nu\rho}\, M_{\mu\sigma} - \eta_{\nu\sigma}\, M_{\mu\rho} - \eta_{\mu\rho}\, M_{\nu\sigma} + \eta_{\mu\sigma}\, M_{\nu\rho})$$

$$[P_\mu, M_{\rho\sigma}] = i\,(\eta_{\mu\rho}\, P_\sigma - \eta_{\mu\sigma}\, P_\rho)$$

$$\left[T^a, T^b\right] = i f^{ab}_{\ \ c}\, T^c \qquad [T^a, P_\mu] = [T^a, M_{\mu\nu}] = 0$$

$$\left[\mathcal{Q}^I_\alpha, P_\mu\right] = \left[\bar{\mathcal{Q}}^I_{\dot\alpha}, P_\mu\right] = 0 \qquad \left[\mathcal{Q}^I_\alpha, T^a\right] = (b_a)^I{}_J\, \mathcal{Q}^J_\alpha \qquad \left[\bar{\mathcal{Q}}^I_{\dot\alpha}, T^a\right] = -\bar{\mathcal{Q}}^J_{\dot\alpha}\, (b_a)_J{}^I$$

$$\left[\mathcal{Q}^I_\alpha, M_{\mu\nu}\right] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha^{\ \beta}\, \mathcal{Q}^I_\beta \qquad \left[\bar{\mathcal{Q}}^I_{\dot\alpha}, M_{\mu\nu}\right] = -\frac{1}{2}\bar{\mathcal{Q}}^I_{\dot\beta}\, (\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}$$

$$\left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^J_{\dot\beta}\right\} = 2\,\delta^{IJ}\,(\sigma^\mu)_{\alpha\dot\beta}\, P_\mu$$

$$\left\{\bar{\mathcal{Q}}^I_{\dot\alpha}, \bar{\mathcal{Q}}^J_{\dot\beta}\right\} = -2\,\epsilon_{\dot\alpha\dot\beta}\, \mathcal{Z}^{IJ\,\dagger} \qquad \left\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\right\} = 2\,\epsilon_{\alpha\beta}\, \mathcal{Z}^{IJ} \quad \text{where} \quad \mathcal{Z}_{IJ} = (a^a_{IJ})\, T^a$$

$$[\mathcal{Z}_{IJ}, \text{anything}] = 0$$

Massless representations

In the light frame $p_\mu = (E, 0, 0, E)$. A state, thus, is determined by its energy and helicity, $|E, \lambda\rangle$.

The eigenvalues of the Pauli-Lubanski vector, $W^\mu = -\frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\; P_\nu\; M_{\rho\sigma}$,

$$W_\mu\; |E, \lambda\rangle = \lambda\; p_\mu\; |E, \lambda\rangle$$

Then, the state $\mathcal{Q}^I_\alpha\; |E, \lambda\rangle$,

$$W_0\; \mathcal{Q}^I_\alpha\; |E, \lambda\rangle = \left(\mathcal{Q}^I_\alpha\; W_0 + \left[W_0, \mathcal{Q}^I_\alpha\right]\right)\; |E, \lambda\rangle = E\left(\lambda\;\delta_\alpha^{\;\beta} - \frac{1}{2}(\sigma^3)_\alpha^{\;\beta}\right)\mathcal{Q}^I_\beta\; |E, \lambda\rangle$$

Thus $\mathcal{Q}^I_1$ lowers the helicity by $1/2$ and $\mathcal{Q}^I_2$ raises it by $1/2$.

Conversely, $\bar{\mathcal{Q}}^I_{\dot{1}}$ raises the helicity by $1/2$ and $\bar{\mathcal{Q}}^I_{\dot{2}}$ lowers it by $1/2$ due to the extra minus factor.

Coming back to the SUSY algebra,

$$\left\{\mathcal{Q}^I_1, \bar{\mathcal{Q}}^J_{\dot{1}}\right\} = 4E\;\delta^{IJ} \qquad \left\{\mathcal{Q}^I_1, \bar{\mathcal{Q}}^J_{\dot{2}}\right\} = \left\{\mathcal{Q}^I_2, \bar{\mathcal{Q}}^J_{\dot{1}}\right\} = \left\{\mathcal{Q}^I_2, \bar{\mathcal{Q}}^J_{\dot{2}}\right\} = 0$$

This means that we can just set $\mathcal{Q}^I_2 = 0 \quad \Rightarrow \quad \mathcal{Z}^{IJ} = 0$

Massless representations

The SUSY algebra reduces to a set

$$a^I := \frac{1}{2\sqrt{E}}\,\mathcal{Q}^I_1 \quad \& \quad a^{J\dagger} := \frac{1}{2\sqrt{E}}\,\bar{\mathcal{Q}}^J_{\dot{1}}$$

of creation/annihilation operators obeying a Clifford algebra,

$$\left\{a^I, a^{J\dagger}\right\} = \delta^{IJ} \qquad \left\{a^I, a^J\right\} = \left\{a^{I\dagger}, a^{J\dagger}\right\} = 0$$

Any irreducible representation is characterized by a *ground state*, $|E, \lambda_0\rangle$,

$$a^I\,|E, \lambda_0\rangle = 0 \qquad \forall I = 1, \ldots, \mathcal{N}$$

We can build the multiplet by acting with the creation operators,

$$a^{I_1\dagger}\,\cdots a^{I_k\dagger}\,|E, \lambda_0\rangle = |E, \lambda_0 + k/2; I_1\,\cdots I_k\rangle \qquad \#_{\text{states}} = \binom{\mathcal{N}}{k}$$

There is a maximum singlet state, $a^{I_1\dagger}\,\cdots a^{I_\mathcal{N}\dagger}\,|E, \lambda_0\rangle$, with helicity $\lambda_0 + \mathcal{N}/2$.

Massless multiplets

The total number of states in a massless multiplet is

$$N_{\rm massless} = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} 1^k \, 1^{\mathcal{N}-k} = (1+1)^{\mathcal{N}} = 2^{\mathcal{N}}$$

There is an equal number of fermions and bosons

$$0 = (1-1)^{\mathcal{N}} = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} (-1)^k \, 1^{\mathcal{N}-k} \quad \Rightarrow \quad \sum_{k=0}^{\mathcal{N}/2} \binom{\mathcal{N}}{2k} = \sum_{k=0}^{\mathcal{N}/2} \binom{\mathcal{N}}{2k+1}$$

$\star$ $\mathcal{N}=2$ vector or chiral multiplet ($\lambda_0 = 0$ and $\lambda_0 = -1$), Ψ, contains

- 2 states with helicities ± 1 (a vector boson A_μ)
- 4 states with helicities $\pm 1/2$ (two Weyl fermions ψ and λ)
- 2 states with helicity 0 (a complex scalar ϕ)

by CPT invariance (Lorentz-covariant QFT). It can be decomposed in terms of $\mathcal{N}=1$ vector $V \equiv (A_\mu, \lambda)$ and chiral $\Phi = (\phi, \psi)$ multiplets.

Massless representations

⋆ $\mathcal{N} = 4$ vector multiplet ($\lambda_0 = -1$) contains

- 2 states with helicities ± 1 (a vector boson A_μ)
- 8 states with helicities $\pm 1/2$ (four Weyl fermions ψ_I)
- 6 states with helicity 0 (six scalars Φ^A)

It can be decomposed in terms of $\mathcal{N} = 2$ vector $\Psi \equiv (A_\mu, \lambda, \psi, \phi)$ and hyper $\mathcal{H} = (\phi_q, \phi_{\tilde{q}}, \psi_q, \psi_{\tilde{q}})$ multiplets.

⋆ $\mathcal{N} = 8$ (maximum) multiplet ($\lambda_0 = -2$), has

- 2 states with helicities ± 2
- 16 states with helicities $\pm 3/2$
- 56 states with helicities ± 1
- 112 states with helicities $\pm 1/2$
- 70 states with helicity 0

These are described by the graviton, $g_{\mu\nu}$, the gravitino,ψ_μ, and a bunch of fermions and scalars that will not play an important rôle in what follows.

Massive representations

In the rest frame, $p_\mu = (m, 0, 0, 0)$. A state, thus, is determined by its mass, its spin and the third component of the spin, $|m, s, s_3\rangle$.

The supercharges are operators of spin $1/2$, thus

$$\mathcal{Q}^I_\alpha \; |m, s, s_3\rangle = \sum_{\tilde{s}_3} c^{(+)}_{s_3 \tilde{s}_3} \; |m, s + 1/2, \tilde{s}_3\rangle + \sum_{\tilde{s}_3} c^{(-)}_{s_3 \tilde{s}_3} \; |m, s - 1/2, \tilde{s}_3\rangle$$

The same is true for $\bar{\mathcal{Q}}^I_{\dot\alpha}$ with, of course, different coefficients.

Coming back to the SUSY algebra,

$$\left\{ \mathcal{Q}^I_1, \bar{\mathcal{Q}}^J_{\dot 1} \right\} = \left\{ \mathcal{Q}^I_2, \bar{\mathcal{Q}}^J_{\dot 2} \right\} = 2m \, \delta^{IJ} \qquad \left\{ \mathcal{Q}^I_1, \bar{\mathcal{Q}}^J_{\dot 2} \right\} = \left\{ \mathcal{Q}^I_2, \bar{\mathcal{Q}}^J_{\dot 1} \right\} = 0$$

and putting for the moment $\mathcal{Z}_{IJ} = 0 \quad \Rightarrow \quad \left\{ \mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta \right\} = \left\{ \bar{\mathcal{Q}}^I_{\dot\alpha}, \bar{\mathcal{Q}}^J_{\dot\beta} \right\} = 0$

Then, we can proceed almost as before by defining

$$a^I_\alpha := \frac{1}{\sqrt{2m}} \, \mathcal{Q}^I_\alpha \quad \& \quad a^{J\dagger}_{\dot\beta} := \frac{1}{\sqrt{2m}} \, \bar{\mathcal{Q}}^J_{\dot\beta}$$

Massive representations

These are creation/annihilation operators obeying a $2\mathcal{N}$-dim Clifford algebra,

$$\left\{a^I_\alpha, a^{J\dagger}_{\dot\beta}\right\} = \delta_{\alpha\dot\beta}\,\delta^{IJ} \qquad \left\{a^I_\alpha, a^J_\beta\right\} = \left\{a^{I\dagger}_{\dot\alpha}, a^{J\dagger}_{\dot\beta}\right\} = 0$$

An irreducible representation is characterized by a *spin multiplet of ground states*, $|m, s_{(0)}, s_3\rangle$,

$$a^I_\alpha\, |m, s_{(0)}, s_3\rangle = 0 \qquad \forall I = 1, \ldots, \mathcal{N} \qquad \alpha = 1, 2$$

We can build the multiplet by acting with the creation operators,

$$a^{I_1\dagger}_{\dot\alpha_1} \cdots a^{I_k\dagger}_{\dot\alpha_k}\, |m, s_{(0)}, s_3\rangle \qquad \#_{\text{states}} = \binom{2\,\mathcal{N}}{k}$$

The states are totally antisymmetric under interchange of $(\dot\alpha_i\, I_i) \leftrightarrow (\dot\alpha_j\, I_j)$.

There is a maximum spin, $s_{\max} = s_{(0)} + \mathcal{N}/2$, and a minimum spin that is $s_{\min} = 0$, if $s_{(0)} \leq \mathcal{N}/2$, or $s_{\min} = s_{(0)} - \mathcal{N}/2$ otherwise.

The *top* state, reached after all $2\mathcal{N}$ operators have been applied, has spin $s_{(0)}$. It is obtained by applying operators $a^{I_k\dagger}_{\dot 1}\, a^{I_k\dagger}_{\dot 2}$ all carrying vanishing spin.

Massive multiplets

A CPT invariant multiplet demands $s_{(0)} = 0$. The story proceeds as before,

$$N_{\text{massive}} = \sum_{k=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{k} = \sum_{k=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{k} 1^k \, 1^{2\mathcal{N}-k} = (1+1)^{2\mathcal{N}} = 2^{2\mathcal{N}}$$

There is still an equal number of fermions and bosons.

This poses a puzzle for the supersymmetric Higgs mechanism:

If we build a Lagrangian with massless fields, as in the Standard Model, they belong to short representations of $2^{\mathcal{N}}$ states.

How can the Higgs mechanism operate?

In a Higgs vacuum, some fields become massive and, thus, belong to a long representation of $2^{2\mathcal{N}}$ states.

A discrete quantity cannot vary continuously: quantum corrections cannot change the length of the multiplet.

Are there massive short multiplets? Yes!

Massive BPS multiplets

Consider $\mathcal{Z}^{IJ} \neq 0$. Since they commute with everything, the central extensions can be diagonalized.

Choose a basis in the representation space where they are represented by the complex numbers z_{IJ}, $z_{IJ} = -z_{JI}$.

By means of U, $\bar{z}_{IJ} = U_I{}^K\, U_J{}^L\, z_{KL}$, they can be brought to the form

$$\bar{z} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \qquad D = \mathrm{diag}\left(z_{(r)}\right) \qquad r = 1, \cdots, \mathcal{N}/2$$

$z_{(r)}$ being real and non-negative. (If $\mathcal{N}$ is odd, additional row of zeros.)

Now redefine the supercharges

$$U^I{}_J\, \mathcal{Q}^J_\alpha \to \mathcal{Q}^I_\alpha \qquad \bar{\mathcal{Q}}^I_{\dot\alpha}\left(U^{-1}\right)_J{}^I \to \bar{\mathcal{Q}}^I_{\dot\alpha}$$

and introduce double indices, $I = (a, r)$, compatible with the form of $\bar{z}$.

The SUSY algebra reads, in this transformed basis,

$$\left\{\mathcal{Q}^{(a,r)}_\alpha, \bar{\mathcal{Q}}^{(b,s)}_{\dot\beta}\right\} = 2\, \delta^{ab}\, \delta^{rs}\, (\sigma^\mu)_{\alpha\dot\beta} P_\mu$$

Massive BPS multiplets

$$\left\{ \mathcal{Q}_\alpha^{(a,r)}, \mathcal{Q}_\beta^{(b,s)} \right\} = 2\, \epsilon_{\dot\alpha\dot\beta}\, \epsilon^{ab}\, \delta^{rs}\, z_{(r)} \qquad \left\{ \bar{\mathcal{Q}}_{\dot\alpha}^{(a,r)}, \bar{\mathcal{Q}}_{\dot\beta}^{(b,s)} \right\} = -2\, \epsilon_{\dot\alpha\dot\beta}\, \epsilon^{ab}\, \delta^{rs}\, z_{(r)}$$

For odd $\mathcal{N}$, we also have

$$\left\{ \mathcal{Q}_\alpha^{\mathcal{N}}, \bar{\mathcal{Q}}_{\dot\beta}^{I} \right\} = 2\, \delta^{\mathcal{N} I}\, (\sigma^\mu)_{\alpha\dot\beta} P_\mu \qquad \left\{ \mathcal{Q}_\alpha^{\mathcal{N}}, \mathcal{Q}_\beta^{I} \right\} = \left\{ \bar{\mathcal{Q}}_{\dot\alpha}^{\mathcal{N}}, \bar{\mathcal{Q}}_{\dot\beta}^{I} \right\} = 0$$

We saw earlier that massless multiplets represent central charges trivially. For the massive case,

$$A_{\alpha r}^{\pm} := \frac{1}{2} \left(\mathcal{Q}_\alpha^{(1,r)} \pm \bar{\mathcal{Q}}^{\dot\alpha(2,r)} \right) \qquad \text{and Hermitian adjoints}$$

Dotted and undotted indices are mixed while preserving covariance.

The SUSY algebra reads, $\left\{ A_{\alpha r}^{\pm}, A_{\beta s}^{\pm} \right\} = \left\{ A_{\alpha r}^{\pm}, A_{\beta s}^{\mp} \right\} = \left\{ A_{\alpha r}^{\pm}, \left(A_{\beta s}^{\mp} \right)^\dagger \right\} = 0,$

$$\left\{ A_{\alpha r}^{\pm}, \left(A_{\beta s}^{\pm} \right)^\dagger \right\} = \delta_{\alpha\beta}\, \delta_{rs}\, (m \pm z_{(r)}) \qquad \Rightarrow \qquad m \geq z_{(r)}$$

The mass is bounded from below by the eigenvalues of the central charges.

Massive BPS multiplets

The massive multiplet saturating the (Bogomol'nyi) bound is special. Assume that it is satisfied for N eigenvalues $z_{(r)}$.

The corresponding $A^{-}_{\alpha r}$ are represented trivially. By rescaling

$$a^{\pm}_{\alpha r} := (m \pm z_{(r)})^{-1/2} \, A^{\pm}_{\alpha r} \qquad a^{\mathcal{N}}_{\alpha} := m^{-1/2} \, \mathcal{Q}^{\mathcal{N}}_{\alpha} \quad \text{(if } \mathcal{N} \text{ is odd)}$$

we end up with a Clifford algebra for $2(\mathcal{N} - N)$ fermionic degrees of freedom!

The situation parallels the one without central charges except for the fact that:

$\mathcal{N}$ is effectively reduced by N, the central charges satisfying the BPS bound.

It is immediate to see that $N_{\max} = \mathcal{N}/2$. In that case, the Clifford algebra ends up being $\mathcal{N}$ dimensional and the corresponding multiplets are short!

This is how the Higgs mechanism operates!

All fields becoming massive due to the Higgs mechanism are BPS states.