Lecture 7: $\mathcal{N} = 2$ supersymmetric gauge theory

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Summary from Lecture 6

We introduced the extended super-Poincaré algebra:

$$\left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^J_{\dot\beta}\right\} = 2\,\delta^{IJ}\,(\sigma^\mu)_{\alpha\dot\beta}P_\mu \qquad \left\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\right\} = 2\,\epsilon_{\alpha\beta}\,\mathcal{Z}^{IJ} \quad \cdots$$

where $\mathcal{Z}_{IJ} = (a^a_{IJ})\,T^a$, and $\left[T^a, T^b\right] = if^{ab}_{\ c}\,T^c$. $I, J, \ldots$ go from 1 to $\mathcal{N}$.

The mass of all states is bounded from below, $m \geq z_{(r)}$.

The number of states in massless and BPS massive multiplets is $2^{\mathcal{N}}$, while that of non-BPS massive multiplets is $2^{2\mathcal{N}}$.

Thus, fields becoming massive due to the Higgs mechanism are BPS.

Exact spectrum

Since they are short multiplets, their spectrum can be safely extended from weak to strong coupling: it is quantum mechanically exact!

R-symmetry

Recall that $\mathcal{N} = 1$ supersymmetry has a global $U(1)$ R-symmetry. It is an automorphism of the algebra acting non-trivially on the supercharges.

If we go back to the extended SUSY algebra, it is immediate to see that, for $\mathcal{N} > 1$, the R-symmetry group is $U(\mathcal{N})$.

The supercharges (fermions) transform in the vector representation, $\mathcal{Q}_\alpha^I$.

For future reference let us recall the R-symmetry group for two cases:

- For $\mathcal{N} = 2$, the R-symmetry group is $U(2) = SU(2)_R \times U(1)_R$.

  In particular, for instance, ψ and λ in the $\mathcal{N} = 2$ vector or chiral multiplet, Ψ belong to a doublet of $SU(2)_R$.

- For $\mathcal{N} = 4$, the R-symmetry group is $U(4) = SU(4)_R \times U(1)_R$ (besides, $SU(4)_R \simeq SO(6)_R$, which happens to be the isometry group of S^5).

$\mathcal{N} = 2$ supersymmetric gauge theory – the superfield

$\mathcal{N} > 1$ is more constrained than $\mathcal{N} = 1$; it is a particular subcase.

The vector superfield Ψ is made of a vector, V, and a chiral, Φ; they belong to the same representation of the non-Abelian gauge group.

The $\mathcal{N} = 2$ action must be simpler because, being enhanced by $SU(2)$, the R-symmetry relates the fermions of both components of Ψ.

In particular, this implies that the superpotential vanishes

$$W(\Phi) = 0$$

Indeed, the $\mathcal{N} = 1$ superfields fit into a chiral $\mathcal{N} = 2$ superfield,

$$\begin{aligned} \Psi(x, \theta, \vartheta) \;&=\; \Phi(z, \theta) + \sqrt{2}\,\vartheta^\alpha \; W_\alpha(z, \theta) \\ &\quad -\frac{1}{2}\vartheta^\alpha\,\vartheta_\alpha \int d^2\bar\theta \; \left[\Phi(z - i\theta\,\sigma\,\bar\theta, \theta, \bar\theta)\right]^\dagger \; e^{-2gV(z - i\theta\,\sigma\,\bar\theta, \theta, \bar\theta)} \end{aligned}$$

where $z^\mu = x^\mu + i\theta\,\sigma^\mu\,\bar\theta + i\vartheta\,\sigma^\mu\,\bar\vartheta = y^\mu + i\vartheta\,\sigma^\mu\,\bar\vartheta$ and $W_\alpha = -\frac{1}{8}\bar{D}^2 e^{-2V} D_\alpha e^{2V}$.

$\mathcal{N} = 2$ supersymmetric gauge theory – the prepotential

The $\mathcal{N} = 2$ superspace notation makes clear that the action is written in terms of a single function, $\mathcal{F}(\Psi)$ called the prepotential

$$\mathcal{S} = \int d^4x\, \mathcal{L} = \frac{1}{16\pi}\, \mathbb{I}\mathrm{m} \left[\int d^4x\, d^2\theta\, d^2\vartheta\, \mathcal{F}(\Psi) \right]$$

The two *remaining* functions in $\mathcal{N} = 1$ theory with adjoint superfields, $f(\Phi)$ and $K(\Phi, \Phi^\dagger)$, can be written in terms of a single holomorphic function. Indeed,

$$f(\Phi) = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^2} \qquad K(\Phi, \Phi^\dagger) = \frac{1}{2i} \left[\Phi^\dagger\, e^{2V}\, \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} - h.c. \right]$$

Being holomorphic, it inherits *good* properties (that you will see later). For instance, $f(\Phi)$ is not corrected after 1-loop; thus $\mathcal{F}(\Psi)$ isn't either.

Formally, the expression above is the most general $\mathcal{N} = 2$ Lagrangian for a supersymmetric gauge theory. In the case of an effective field theory, $\mathcal{F}$ is not restricted to be quadratic. It is only constrained by holomorphicity.

$\mathcal{N} = 2$ supersymmetric gauge theory – the field content

At tree-level, the prepotential reads

$$\mathcal{F}_{\rm class}(\Psi) := \mathcal{F}_0(\Psi) = \frac{1}{2}\tau_0 \,\mathrm{Tr}\, \Psi^2$$

which has the right scaling dimension.

If we plug it into $\mathcal{S}$ and integrate on ϑ,

$$\mathcal{L} = \frac{1}{16\pi} \mathbb{I}\mathrm{m} \left[\left(\int d^2\theta \, \frac{\partial^2 \mathcal{F}_0(\Phi)}{\partial \Phi^a \partial \Phi^b} \, W^{a\alpha} \, W^b{}_\alpha + 2 \int d^2\theta \, d^2\bar{\theta} \, \left(\Phi^\dagger e^{2gV} \right)^a \frac{\partial \mathcal{F}_0}{\partial \Phi^a} \right) \right]$$

where the superfields are in the adjoint of $SU(N_c)$, $\Phi = \Phi^a \, T^a$, $W_\alpha = W_\alpha^a \, T^a$, *i.e.*, $a, b, \ldots$ are Lie algebra indices and $\mathrm{Tr}\,(T^a \, T^b) = \delta^{ab}$.

Ψ is decomposed in $V \equiv (A_\mu, \lambda)$ and $\Phi = (\phi, \psi)$. We refer to ϕ as the higgs, the Weyl spinors ψ and λ as the higgsino and the gluino, and A_μ as the gluon.

Therefore, this is just a particular non-Abelian gauge theory with scalars and fermions in the adjoint, and certain couplings dictated by SUSY.

$\mathcal{N} = 2$ supersymmetric gauge theory – the Lagrangian

The previous $\mathcal{N} = 2$ Lagrangian can be written in terms of its components

$$\mathcal{L} = \frac{1}{g_0^2}\,\mathrm{Tr}\left(-\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + g_0^2\,\frac{\theta_0}{32\pi^2}\,F_{\mu\nu}\,\widetilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger\,D^\mu\phi - \frac{1}{2}\,[\phi^\dagger,\phi]^2 \right.$$
$$\left. - i\,\lambda\sigma^\mu D_\mu\bar{\lambda} - i\,\bar{\psi}\bar{\sigma}^\mu D_\mu\psi - i\sqrt{2}\,[\lambda,\psi]\,\phi^\dagger - i\sqrt{2}\,[\bar{\lambda},\bar{\psi}]\,\phi \right)$$

where we used the expression for the *bare* complexified coupling constant

$$\tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}$$

The quadratic prepotential gives the renormalizable *microscopic* Lagrangian.

The theory is asymptotically free

$$\beta(g) = \mu\,\frac{dg}{d\mu} = -\frac{g^3}{16\pi}\left(\frac{11}{3} - \frac{2}{3}\times 2 - \frac{1}{3}\right)\,C_2(G) = -\frac{g^3}{16\pi}\,(2\,N_c)$$

hence, confinement is expected to be present at strong coupling.

$\mathcal{N} = 2$ supersymmetric gauge theory – R-symmetry and chiral anomaly

This theory has a classical global R-symmetry $U(2)_R = SU(2)_R \times U(1)_R$:

- The $SU(2)_R$ rotates the two supercharges as well as the two fermions as doublets. The gauge boson and the higgs are singlets.
- The $U(1)_R$ acts as $\Phi \to e^{2i\alpha}\,\Phi(e^{-i\alpha}\theta)$ and $V \to V(e^{-i\alpha}\theta)$. Since $\theta \to e^{i\alpha}\theta$, $\vartheta \to e^{i\alpha}\vartheta$, $\mathcal{F}$ must have charge 4, *i.e.*, $\mathcal{F} \to e^{4i\alpha}\,\mathcal{F}$.

$U(1)_R$ acts as a chiral symmetry. Thus, it is broken quantum mechanically

$$\partial_\mu\, j_R^\mu = -\frac{N_c}{8\pi^2}\,\mathrm{Tr}\, F_{\mu\nu}\,\widetilde{F}^{\mu\nu}$$

This dictates the way the 1-loop effective action changes under $U(1)_R$

$$\delta\mathcal{L}_{\mathrm{eff}} = \alpha\,\partial_\mu\, j_R^\mu = -\frac{\alpha\, N_c}{8\pi^2}\,\mathrm{Tr}\, F_{\mu\nu}\,\widetilde{F}^{\mu\nu}$$

due to Noether's theorem.

$\mathcal{N} = 2$ supersymmetric gauge theory – full perturbative prepotential

Seiberg's non-renormalization theorem (to be discussed) applies:

$$\mathcal{F}_{\rm pert}(\Phi) = \mathcal{F}_{\rm class}(\Phi) + \mathcal{F}_{1-{\rm loop}}(\Phi)$$

gives the complete Lagrangian to all orders in perturbation theory.

This implies that the full perturbative contribution to the effective action can be obtained by integrating the infinitesimal anomalous variation.

Notice that $\log \mathrm{Tr}\,\Phi^2 \rightarrow \log \mathrm{Tr}\,\Phi^2 + 4i\alpha$ under $U(1)_R$.

At the level of the prepotential, thus, the result is simply

$$\mathcal{F}_{\rm pert}(\Phi) = \frac{1}{2}\tau_0\,\mathrm{Tr}\,\Phi^2 + \frac{i\,N_c}{4\pi}\,\mathrm{Tr}\,\Phi^2\,\log\left(\frac{\mathrm{Tr}\,\Phi^2}{\Lambda^2}\right)$$

the coefficient being fixed from the value of the chiral anomaly.

Λ is the quantum mechanical dynamically generated scale, as in QCD.

$\mathcal{N} = 2$ supersymmetric gauge theory – the fate of the $U(1)_R$

The only term in $\mathcal{L}_{\rm eff}$ affected by the $U(1)_R$ transformation is the θ-term,

$$\theta \rightarrow \theta + 4N_c\,\alpha$$

since, recall

$$\delta\mathcal{L}_{\rm eff} = \alpha\,\partial_\mu\, j_R^\mu = -\frac{\alpha\,N_c}{8\pi^2}\,{\rm Tr}\,F_{\mu\nu}\,\widetilde{F}^{\mu\nu}$$

At the level of the effective action however, notice that, since

$$\frac{1}{32\pi^2}\int F_{\mu\nu}\widetilde{F}^{\mu\nu} \in \mathbb{Z} \qquad \text{(instanton number)}$$

a discrete subgroup, $\mathbb{Z}_{4N_c} \subset U(1)_R$, remains a symmetry of the perturbative effective action,

$$\alpha = 2\pi i\,\frac{k}{4N_c} \qquad k = 1, \cdots, 4N_c$$

This is a general feature of the $U(1)_R$ subgroup of $U(\mathcal{N})_R$.

The classical moduli space, $\mathcal{M}_0$

Let us now focus on the classical vacua of the theory. The potential reads

$$V = \frac{1}{2g^2}\,\mathrm{Tr}\left([\phi^\dagger, \phi]^2\right)$$

Unbroken supersymmetry requires that V vanishes in the vacuum.

There is a family of vacua parameterized by constant fields ϕ_0 such that ϕ_0 and $\phi_0^\dagger$ commute. One can always rotate it into the Cartan subalgebra, $\mathcal{H}$,

$$\langle\phi\rangle = \phi_0 = \sum_{k=1}^{N_c-1} a_k\, H^k$$

H^k are the generators of $\mathcal{H}$ and a_k are complex numbers. For $\mathcal{G} = su(2)$, the Cartan subalgebra is simply generated by the Pauli matrix σ_3 and $\langle\phi\rangle = a\,\sigma_3$.

For $a \neq 0$, the gauge symmetry group is broken, $SU(2) \to U(1)$.

There is a *singular* point, $a = 0$, where the gauge symmetry is unbroken.

The classical moduli space, $\mathcal{M}_0$

ϕ_0 labels a continuous family of inequivalent ground states that constitute the classical moduli space, $\mathcal{M}_0$.

The variables a_k are not gauge invariant, in particular under discrete Weyl transformations. Hence they are not faithfull coordinates for $\mathcal{M}_0$.

The Weyl group for a Lie algebra is generated by reflections in the roots; in the $su(2)$ case, $a\,\sigma_3 \to -a\,\sigma_3$. In general, it acts by conjugation, $\phi_0 \to g^{-1}\phi_0 g$.

Thus, Weyl invariants can be obtained from the characteristic polynomial

$$P_{N_c}(\lambda) = \det(\lambda - \phi_0)$$

The coefficients of the polynomial (sometimes called $W_{A_{N_c-1}}$),

$$P_{N_c}(\lambda) = \lambda^{N_c} - \bar{u}_2(a_k)\,\lambda^{N_c-2} - \bar{u}_3(a_k)\,\lambda^{N_c-3} - \cdots - \bar{u}_{N_c}(a_k)$$

label gauge *inequivalent* vacua and, thus, parameterize faithfully $\mathcal{M}_0$.

There is no λ^{N_c-1} term due to the traceless condition of $su(N_c)$.

The classical moduli space, $\mathcal{M}_0$

A simple calculation shows that

$$\bar{u}_2(a_k) = \frac{1}{2}\,\mathrm{Tr}\,{\phi_0}^2 \qquad \bar{u}_3(a_k) = \frac{1}{3}\,\mathrm{Tr}\,{\phi_0}^3 \qquad \bar{u}_4(a_k) = \frac{1}{4}\,\mathrm{Tr}\,{\phi_0}^4 - \frac{1}{8}\,(\mathrm{Tr}\,{\phi_0}^2)^2$$

These are Casimir operators and, hence, Weyl invariant by construction,

$$\bar{u}_k(a_i) = \frac{1}{k}\,\mathrm{Tr}\,{\phi_0}^k + \text{lower dimensional Casimirs}$$

Coming back to $SU(2)$, there is a single (quadratic) Casimir (call $\bar{u} := \bar{u}_2$),

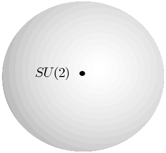
$$\bar{u}(a) = \frac{1}{2}\,\mathrm{Tr}\,{\phi_0}^2 = \frac{1}{2}\mathrm{Tr}\,(a\,\sigma_3)^2 = a^2$$

and the characteristic polynomial reads simply $P_2(\lambda) = \lambda^2 - \bar{u}$.

For $\bar{u} \neq 0$, the gauge symmetry group is broken, $SU(2) \to U(1)$.

There is a singular point, $\bar{u} = 0$, where the gauge symmetry is unbroken.

The classical moduli space, $\mathcal{M}_0$



$\mathcal{M}_0(SU(2))$

Figure: $\mathcal{M}_0$ for $SU(2)$ is the complex $\bar{u}$-plane. The origin displays classical symmetry enhancement. Since $P_2(\lambda) = (\lambda - \bar{u}^{1/2})(\lambda + \bar{u}^{1/2})$, this is captured by the vanishing locus, Σ_0, of the classical discriminant $\Delta_0 := (\bar{u}^{1/2} - (-\bar{u}^{1/2}))^2 = 4\,\bar{u}$.

Another way to define the $\bar{u}_k$ is through the *classical Miura transformation*. Namely, factorizing the characteristic polynomial

$$P_{N_c}(\lambda) = \prod_{i=1}^{N_c} (\lambda - e_i(a^k))$$

where $e_i(a^k)$, $i = 1, \ldots, N_c$, are the eigenvalues of ϕ_0 .

The classical moduli space, $\mathcal{M}_0$

Expanding $P_{N_c}(\lambda)$, it is easy to see that

$$\bar{u}_k(a) = (-1)^{k+1} \sum_{j_1 \neq \cdots \neq j_k} e_{j_1}(a) \cdots e_{j_k}(a)$$

which are symmetric polynomials of the eigenvalues $e_i(a^k)$, thus manifestly invariant under the Weyl group (which acts by permutation).

Consider the $SU(3)$ case (call $\bar{u} := \bar{u}_2$ and $\bar{v} := \bar{u}_3$),

$$P_3(\lambda) = \lambda^3 - \bar{u}\,\lambda - \bar{v} = (\lambda - e_1)\,(\lambda - e_2)\,(\lambda - e_3)$$

where, of course, $\bar{u} = \bar{u}(a)$, $\bar{v} = \bar{v}(a)$,

$$\bar{u}(a) = {a_1}^2 + {a_2}^2 - a_1\,a_2$$

$$\bar{v}(a) = a_1\,a_2\,(a_1 - a_2)$$

and $e_1(a) = a_1$, $e_2(a) = -a_2$ and $e_3(a) = a_2 - a_1$.

The classical moduli space, $\mathcal{M}_0$

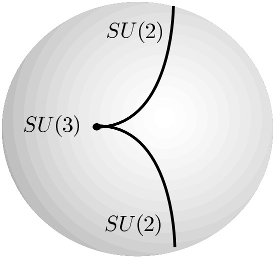


Figure: $\mathcal{M}_0$ for $SU(3)$ has two complex coordinates, $\bar{u}$ and $\bar{v}$. Classical symmetry enhancement is captured by the vanishing locus, Σ_0, of the classical discriminant.

$$\Delta_0 := (e_2 - e_1)^2\,(e_3 - e_2)^2\,(e_1 - e_3)^2 = 4\,\bar{u}^{\,3} - 27\,\bar{v}^{\,2}$$

We will focus from now on almost exclusively in the $SU(2)$ case.

The effective prepotential: perturbative part

Let us consider for the moment a generic situation, *i.e.*, a vacuum $\phi_0 = a\,\sigma_3$ where $SU(2) \to U(1)$. If we evaluate $\mathcal{F}_{\rm class}(a)$,

$$\mathcal{F}_{\rm class}(a) = \frac{1}{2}\tau_0 \,\mathrm{Tr}\, {\phi_0}^2 = \frac{1}{2}\tau_0\, a\, a\, \mathrm{Tr}\,(\sigma_3\, \sigma_3) = \tau_0\, a^2$$

It is convenient, to make contact with the $SU(N_c)$ case,

$$\mathcal{F}_{\rm class}(a_k) = \frac{1}{2N_c}\tau_0 \sum_{\alpha_+ \in \Delta_+} z^2_{\alpha_+}$$

where Δ_+ are the positive roots, and $z_{\alpha_+} := \alpha_+ \cdot \phi_0 = z_{\alpha_+}(a_k)$.

In the case of $SU(2)$, $z_a := 2a$, and

$$\mathcal{F}_{\rm class}(a) = \frac{1}{4}\,\tau_0\, {z_a}^2 = \tau_0\, \bar{u}$$

The 1-loop correction can be obtained by plugging ϕ_0 in the expression above

$$\mathcal{F}_{1-{\rm loop}}(a) = \frac{i}{4\pi}\, {z_a}^2\, \log\left(\frac{{z_a}^2}{\Lambda^2}\right) = \frac{i}{\pi}\, \bar{u}\, \log\left(\frac{4\bar{u}}{\Lambda^2}\right)$$

The effective prepotential: perturbative part

Again, this can be similarly written in the $SU(N_c)$ case as

$$\mathcal{F}_{1-\mathrm{loop}}(a_k) = \frac{i}{4\pi} \sum_{\alpha_+ \in \Delta_+} z_{\alpha_+}^2 \log\left(\frac{z_{\alpha_+}^2}{\Lambda^2}\right)$$

Notice that $\mathcal{F}_{1-\mathrm{loop}}(a_k)$ diverges if $z_{\alpha_+}(a_k)$ vanishes. These singularities are in one-to-one correspondence to those of $\mathcal{M}_0$.

From the kinetic term of the complex scalar, we see that

$$\begin{aligned}\mathrm{Tr}\,(D_\mu \phi_0)^\dagger\, D^\mu \phi_0 &= \mathrm{Tr}\,(i[W_\mu, \phi_0])^\dagger\,(i[W_\mu, \phi_0]) \\ &= \frac{1}{2} \sum_{\alpha_+ \in \Delta_+} |z_{\alpha_+}|^2\, W_\mu^{\pm\alpha_+}\, W^{\mu\,\pm\alpha_+}\end{aligned}$$

Whenever $z_{\alpha_+}(a_k) = 0$:

- A pair of charged gauge bosons become massless, and
- Classically, there is gauge symmetry enhancement.