Lecture 8: The quantum moduli space

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The effective prepotential: perturbative part

Let us consider for the moment a generic situation, *i.e.*, a vacuum $\phi_0 = a\,\sigma_3$ where $SU(2) \to U(1)$. If we evaluate $\mathcal{F}_{\rm class}(a)$,

$$\mathcal{F}_{\rm class}(a) = \frac{1}{2}\tau_0\, \mathrm{Tr}\, {\phi_0}^2 = \frac{1}{2}\tau_0\, a\, a\, \mathrm{Tr}\,(\sigma_3\, \sigma_3) = \frac{1}{2}\tau_0\, a^2$$

It is convenient, to make contact with the $SU(N_c)$ case, to write

$$\mathcal{F}_{\rm class}(a) = \frac{1}{4}\,\tau_0\, {z_a}^2 = \tau_0\, \bar{u}$$

where, in this case, $z_a := 2a$.

The 1-loop correction can be obtained by plugging ϕ_0 in the expression above

$$\mathcal{F}_{1-\rm loop}(a) = \frac{i}{4\pi}\, {z_a}^2\, \log\left(\frac{{z_a}^2}{\Lambda^2}\right) = \frac{i}{\pi}\, \bar{u}\, \log\left(\frac{4\bar{u}}{\Lambda^2}\right)$$

Notice that $\mathcal{F}_{1-\rm loop}(a)$ diverges if z_a (or, equivalently, $\bar{u}$) vanishes. In general, these singularities are in one-to-one correspondence to those of $\mathcal{M}_0$.

Massive charged gauge bosons are BPS

Indeed, whenever this happens there is a pair of charged gauge bosons that become massless; classically, there is gauge symmetry enhancement.

We know that the charged gauge bosons must saturate the BPS bound. In fact, from the kinetic term of the complex scalar, we see that

$$(D_\mu \phi_0)^\dagger D^\mu \phi_0 = (i[W_\mu, \phi_0])^\dagger (i[W_\mu, \phi_0]) = W_\mu^a W^{\mu\, b}\, a\, a\, (\mathrm{ad}_a \sigma_3, \mathrm{ad}_b \sigma_3)$$

Then, when taking the trace

$$\mathrm{Tr}\, (D_\mu \phi_0)^\dagger D^\mu \phi_0 \simeq {z_a}^2\, W_\mu^+ W^{\mu\,+} + {z_a}^2\, W_\mu^- W^{\mu\,-}$$

the gauge bosons associated to the step generators $\sigma_\pm$ get a (BPS) mass

$$M_\pm(a) = \frac{1}{2}\, |z_a| = |a|$$

According to our earlier discussion z_a must be the value of the central charge corresponding to a gauge boson $W_\mu^\pm$.

Massive charged gauge bosons are BPS

In order to see how this happens, let us consider a simpler example: the $3d$ $\mathcal{N}=2$ Abelian Higgs model,

$$\mathcal{L}_{\mathcal{N}=2} = \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}|D_\mu\varphi|^2 - \frac{e^2}{8}(|\varphi|^2 - {\varphi_0}^2)^2 \right\}$$

where, for simplicity, we consistently set to zero a scalar field and all fermions.

Noether $\mathcal{N}=2$ supercharges are given by

$$\mathcal{Q} = \sqrt{2} \int d^2x \, \mathcal{J}^0_{\mathcal{N}=2} := \overline{\eta}\, Q + \overline{Q}\, \eta$$

where $\mathcal{J}^\mu_{N=2}$ is the current associated with SUSY transformations. By explicit computation (we neglect quadratic terms in the fermions)

$$Q = \sqrt{2} \int d^2x \left[\left(-\frac{1}{2}\, \epsilon^{\mu\nu\lambda}\, F_{\mu\nu}\, \gamma_\lambda - \frac{e}{2}(|\varphi|^2 - {\varphi_0}^2) \right) \gamma^0 \Sigma + i(\not{D}\varphi)^* \gamma^0 \psi \right]$$

For static configurations with $A_0 = 0$, we compute the canonical brackets,

Massive charged gauge bosons are BPS

$$\left\{Q_\alpha, \overline{Q}_\beta\right\} = 2\,(\gamma_0)_{\alpha\beta}\; P^0 + \delta_{\alpha\beta}\; z$$

where the mass reads

$$P^0 = E = \int d^2x\;\left[\frac{1}{4}\,F_{ij}^2 + \frac{1}{2}\,|D_i\varphi|^2 + \frac{e^2}{8}(|\varphi|^2 - {\varphi_0}^2)^2\right]$$

while the central charge is given by

$$z = -\int d^2x\;\left[\frac{e}{2}\,\epsilon^{ij}\,F_{ij}\,(|\varphi|^2 - {\varphi_0}^2) + i\epsilon^{ij}\,(D_i\varphi)\,(D_j\varphi)^*\right]$$

It is the topologically quantized magnetic flux of the Abelian Higgs model,

$$z = \int d^2x\;\epsilon^{ij}\,\partial_i\left(e\,{\varphi_0}^2\,A_j + i\;\varphi^*\,D_j\varphi\right) = e\,{\varphi_0}^2 \oint A_i\,dx^i = 2\pi\,n\;{\varphi_0}^2$$

after Stokes theorem (using $D_i\varphi \to 0$ at ∞, where $\varphi \to \varphi_0\,e^{in\theta}$). $n \in \mathbb{Z}$ gives the homotopy class to which A_i belongs.

Massive charged gauge bosons are BPS

The $\mathcal{N} = 2$ Abelian Higgs model has taught us that magnetic BPS states have

$$M(n) = |z(n)| \simeq n\, {\varphi_0}^2$$

The same exercise can be done in the $\mathcal{N} = 2$ supersymmetric gauge theory.

For a generic vacuum, $SU(2) \to U(1)$, the central charge of a state with n (m) units of electric (magnetic) charge with respect to the unbroken $U(1)$ reads

$$Z(n, m) = n\, a + m\, \tau_0\, a$$

It arises, as before, from boundary terms in the supercharge algebra.

From $M_\pm(a) = |a|$ we can read off the electric charges of the massive gauge bosons with respect to the unbroken $U(1)$.

A hypothetical ('t Hooft–Polyakov) magnetic monopole should have a mass

$$M_{\rm monopole}(a) = |\tau_0\, a| \simeq \frac{4\pi}{g^2} |a|$$

The rôle of instantons

Recall that the theory is asymptotically free,

$$\beta(g) = \mu \frac{dg}{d\mu} = -\frac{g^3}{4\pi}$$

which can be written as

$$\frac{d\, \mathbb{I}\mathrm{m}\, \tau}{d \log \mu} = \frac{2}{\pi} \qquad \Rightarrow \qquad \mathbb{I}\mathrm{m}\, \tau = \frac{4\pi}{g^2} = \frac{2}{\pi} \log\left(\frac{\mu}{\Lambda}\right)$$

The energy scale to compute the Wilsonian effective action is provided by the gauge boson mass, a, at a given vacuum, *i.e.*, $\mu \sim a$.

In the high momenta region, $p \sim a \gg \Lambda$, the perturbative calculation is reliable.

At lower energies, non-perturbative instanton corrections start to dominate.

A configuration with instanton number k contributes to the path integral as

$$\exp\left(-\frac{8\pi^2 k}{g^2}\right)$$

The full Wilsonian effective action

$$\exp\left(-\frac{8\pi^2 k}{g^2}\right) = \exp\left(-2\pi k\,\frac{2}{\pi}\log\left(\frac{a}{\Lambda}\right)\right) = \left(\frac{\Lambda}{a}\right)^{4\,k}$$

Notice that Λ is defined as the value where formaly g^2 diverges, and hence signals the onset of non-perturbative dominated phenomena for $a \ll \Lambda$.

The most general form of the Wilsonian effective prepotential is

$$\mathcal{F}(a) = \frac{1}{4}\,\tau_0\,z_a^2 + \frac{i}{4\pi}\,z_a^2\,\log\left(\frac{z_a^2}{\Lambda^2}\right) + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\mathcal{F}_k(a)\,\Lambda^{4\,k}$$

$\mathcal{F}_k(a)$ are homogeneous functions of degree $2-4\,k \;\Rightarrow\; \mathcal{F}(a)$ has degree 2.

Computing $\mathcal{F}_k(a)$ means fully solving the Wilsonian effective action for $\mathcal{N}=2$ SYM theory.

This is what Nathan Seiberg and Edward Witten did in 1994!

Symmetries

Getting $\mathcal{F}_k(a)$ as the result of a honest computation is really hard.

Instead, a clever route to construct an effective action amounts to looking first at the set of possible answers.

It is in this respect that the use of symmetries at hand is instrumental.

The resulting action has to be $\mathcal{N} = 2$ supersymmetric (thus, being written in terms of an effective prepotential), possess a generic $U(1)$ gauge group, and (a residual chiral) $\mathbb{Z}_8$ discret symmetry.

Moreover, being a low energy effective action we will content ourselves with terms with, at most, two derivatives. Higher derivative (non-renormalizable) terms are, indeed, present and subleading.

At a generic point of $\mathcal{M}_0$, the Wilsonian effective action should look like

$$\mathcal{L} = \frac{1}{4\pi}\mathbb{I}\mathrm{m}\left[\int d^4\theta \left(\frac{\partial\mathcal{F}(a)}{\partial a}\,\bar{a}\right) + \int d^2\theta\,\frac{1}{2}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\,W^\alpha\,W_\alpha\right)\right]$$

Towards the quantum moduli space

The prepotential $\mathcal{F}(a)$ describes the geometry of the moduli space. In fact, writing down the bosonic kinetic part of the previous Lagrangian,

$$\mathcal{L}_{kin}^{B} = \frac{1}{4\pi}\,\mathbb{I}\mathrm{m}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\right)\left[(\nabla_\mu a)^*\,(\nabla^\mu a) - \frac{1}{4}(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}\tilde{F}^{\mu\nu})\right]$$

it is clear that a is the coordinate of a *manifold* whose metric is given by the imaginary part of the complexified coupling constant, $g_{a\bar{a}}(a) = \mathrm{Im}\,\tau(a)$,

$$ds^2 = g_{a\bar{a}}(a)\,da\,d\bar{a} = \mathrm{Im}\,\tau(a)\,da\,d\bar{a} := \mathbb{I}\mathrm{m}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\right)da\,d\bar{a}$$

usually referred to as the *Zamolodchikov metric*.

Since $\mathcal{F}(a)$ is holomorphic, $\mathbb{I}\mathrm{m}\,\tau(a)$ is an harmonic function and, therefore, it cannot have a global minimum. If the coupling is globally defined, it could not be positive everywhere (unless $\tau(a) = \tau_0$).

$\tau(a)$ cannot be globally defined since $\mathbb{I}\mathrm{m}\,\tau(a) \geq 0$

Towards the quantum moduli space

We should rely on distinct local descriptions valid for different regions of the quantum moduli space.

Whenever $\mathbb{Im}\,\tau(a) \to 0$ there is the need to use a different set of coordinates, $\hat{a}$, such that $\mathbb{Im}\,\hat{\tau}(\hat{a})$ is non-singular and non-vanishing.

This is possible if the singularity of $g_{a\bar{a}}(a)$ is only a coordinate singularity.

Notice that, to all orders in perturbation theory

$$\tau(a) = \tau(\bar{u}(a)) = \tilde{\tau}_0 + \frac{2i}{\pi} \log\left(\frac{\bar{u}(a)}{\Lambda^2}\right) = \frac{i}{\pi}\left[3 + \log\left(\frac{a^2}{\Lambda^2}\right)\right]$$

The logarithm appearing at 1-loop makes $\tau(a)$ a multi-valued function.

Its imaginary part, however, is single-valued and positive in the *semiclassical region* $|a| \gg \Lambda$ (where the expression is valid).

Thus, $\bar{u}(a) = a^2$ is a good local coordinate in the semiclassical patch of the quantum moduli space.

Duality

$\tau(a)$ gives the *electromagnetic* coupling to all orders in perturbation theory.

As mentioned, a and $\bar{a}$ (actually a^2 and $\bar{a}^2$) are good and faithful coordinates in the semiclassical region $a^2 \gg \Lambda^2$.

This means that the original superfields Φ and W_α are the relevant degrees of freedom; the appropriate fields to describe the low-energy effective action.

Recall the form of the *effective* action

$$\mathcal{L} = \frac{1}{4\pi}\mathbb{I}\mathrm{m}\left[\int d^4\theta\ \bar{a}\ \mathcal{F}'(a) + \frac{1}{2}\int d^2\theta\ \mathcal{F}''(a)\, W^\alpha\, W_\alpha\right]$$

Let us define a *dual* field a_D, $a_D := \mathcal{F}'(a)$ (*i.e.*, in the *microscopic theory*, $\Phi_D = \mathcal{F}'(\Phi)$ or, better, $\Phi_D^c = \partial\mathcal{F}(\Phi)/\partial\Phi_c$).

Let us also introduce a *dual* prepotential $\mathcal{F}_D(a_D)$, $\mathcal{F}_D{}'(a_D) := -a$.

This is a Legendre transformation.

Duality

It can be used to show that

$$\mathbb{I}\mathrm{m} \int d^4\theta \; \bar{a} \; \mathcal{F}'(a) = -\mathbb{I}\mathrm{m} \int d^4\theta \; a_D \; \bar{\mathcal{F}}_D{}'(\bar{a}_D) = \mathbb{I}\mathrm{m} \int d^4\theta \; \bar{a}_D \; \mathcal{F}_D{}'(a_D)$$

Thus, the first term in the action is *duality* invariant.

This is reminiscent of a canonical transformation in which $\mathcal{F}'(a)$ resembles a (complex) momentum.

As such, it is a transformation with a *trivial* Jacobian for the integration measure of the path integral.

Concerning the second term, it contains W_α and we have not specified its transformation properties. W_α contains the $U(1)$ field strength $F_{\mu\nu}$,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V = -i\lambda_\alpha + \theta_\alpha \, D - i(\sigma^{\mu\nu}\theta)_\alpha \, F_{\mu\nu} - \theta^2(\sigma^\mu D_\mu \bar{\lambda})_\alpha$$

The $F_{\mu\nu}$ is not arbitrary but of the form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for some A_μ.

Duality

This translates into the Bianchi identity $\partial_\nu \tilde{F}^{\mu\nu} = 0$. This constraint reads, in superspace, $\mathbb{I}\mathrm{m}(D_\alpha W^\alpha) = 0$.

In the path integral we can integrate over V only or, else, over W_α while imposing the constraint by a real Lagrange multiplier which we call V_D,

$$\int \mathcal{D}V \exp\left[\frac{i}{8\pi}\mathbb{I}\mathrm{m}\int d^4x d^2\theta\, \mathcal{F}''(a)\, W^\alpha\, W_\alpha\right] \simeq \int \mathcal{D}W_\alpha \mathcal{D}V_D$$

$$\exp\left[\frac{i}{8\pi}\mathbb{I}\mathrm{m}\int d^4x \left(\int d^2\theta\, \mathcal{F}''(a)\, W^\alpha W_\alpha + \frac{1}{2}\int d^2\theta\, d^2\bar{\theta}\, V_D\, D_\alpha W^\alpha\right)\right]$$

Now, observe that (using $\bar{D}_{\dot{\beta}}\, W^\alpha = 0$)

$$\int d^2\theta\, d^2\bar{\theta}\, V_D\, D_\alpha W^\alpha = -\int d^2\theta\, d^2\bar{\theta}\, D_\alpha V_D\, W^\alpha = \int d^2\theta\, \bar{D}^2(D_\alpha V_D\, W^\alpha)$$

$$= \int d^2\theta\, (\bar{D}^2 D_\alpha V_D)\, W^\alpha = -4\int d^2\theta\, (W_D)_\alpha\, W^\alpha$$

Duality

The path integral becomes Gaussian in W_α

$$\int \mathcal{D}W_\alpha \mathcal{D}V_D \exp\left[\frac{i}{8\pi}\mathbb{I}\mathrm{m}\int d^4x \int d^2\theta \left(\mathcal{F}''(a)\, W^\alpha W_\alpha - 2(W_D)_\alpha\, W^\alpha\right)\right]$$

thus leading to the following expression

$$\int \mathcal{D}V_D \exp\left[\frac{i}{8\pi}\mathbb{I}\mathrm{m}\int d^4x \int d^2\theta \left(-\frac{1}{\mathcal{F}''(a)}\, W_D^\alpha\, W_{D\alpha}\right)\right]$$

This is a remarkable result. The SUSY Yang-Mills action is reexpressed in terms of an almost identical dual action except for the fact that

$$\tau(a) \to -\frac{1}{\tau(a)} \qquad\qquad \tau(a) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)}$$

Thus, the duality transformation inverts the gauge coupling. It is a so-called strong-weak duality. $W_{D\alpha}$ has among its components the dual field strength $\tilde{F}_{\mu\nu}$. This generalizes the old *Montonen-Olive electromagnetic duality*.

The duality group

If we wish to express everything in terms of dual variables,

$$\mathcal{F}_D''(a_D) = -a'(a_D) = -\frac{1}{a_D'(a)} = -\frac{1}{\mathcal{F}''(a)}$$

and the whole Lagrangian keeps its form but now in terms of dual variables

$$\mathcal{L} = \frac{1}{4\pi}\mathbb{I}\mathrm{m}\left[\int d^4\theta\, \bar{a}_D\, \mathcal{F}_D'(a_D) + \frac{1}{2}\int d^2\theta\, \mathcal{F}_D''(a_D)\, W_D^\alpha\, W_{D\alpha}\right]$$

The duality map is a strong/weak coupling transformation and the Lagrangian *remains invariant* formally in terms of the new variables.

It is convenient to rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{8\pi}\mathbb{I}\mathrm{m}\int d^2\theta\, a_D'(a)\, W^\alpha\, W_\alpha + \frac{1}{8\pi i}\int d^4\theta\, (\bar{a}\, a_D - \bar{a}_D\, a)$$

The duality discussed above can be casted as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

The duality group

An extra symmetry can be readily identified in

$$\mathcal{L} = \frac{1}{8\pi}\mathbb{I}\mathrm{m} \int d^2\theta \; a_D{}'(a)\, W^\alpha \, W_\alpha + \frac{1}{8\pi i} \int d^4\theta \; (\bar{a}\, a_D - \bar{a}_D\, a)$$

It is given by the following transformation

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad k \in \mathbb{Z}$$

due to the instanton density and reality properties of the Lagrangian.

Both symmetries, together, generate the group $S\ell(2,\mathbb{Z})$.

The discussion can be extended to $SU(N_c)$, giving $S\ell(2(N_c - 1),\mathbb{Z})$.

The metric on the moduli space can be written in an $S\ell(2,\mathbb{Z})$ invariant fashion

$$ds^2 = \mathbb{I}\mathrm{m}\,(\mathcal{F}''(a))\; da\, d\bar{a} = \mathbb{I}\mathrm{m}\,(da_D\, d\bar{a}) \;= \frac{i}{2}\,(da\, d\bar{a}_D - da_D\, d\bar{a})$$

More on the BPS spectrum

Recall that, for a generic vacuum, the central charge of a dyon configuration or state with n (m) units of electric (magnetic) charge reads

$$Z(n,m) = n\,a + m\,\tau_0\,a$$

This was obtained from a classical computation using Poisson brackets. The right quantum mechanical formula should be duality invariant

$$Z(n,m) = n\,a + m\,a_D = (m \quad n)\begin{pmatrix} a_D \\ a \end{pmatrix}$$

A matrix $\mathbb{M} \in S\ell(2,\mathbb{Z})$ reads

$$\mathbb{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with} \quad a,b,c,d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1$$

A duality transformation of the (a_D, a) coordinates implies an analogous transformation of the charge vector (m, n) by acting with $\mathbb{M}$, the dual quantum numbers, $(\tilde{m}, \tilde{n})$, being still integers.

Browsing the moduli space

The classical moduli space, $\mathcal{M}_0$, has a faithful complex coordinate,

$$\bar{u} = \frac{1}{2}\,\mathrm{Tr}\,\langle\phi\rangle^2 = a^2$$

Now, the gauge invariant object relevant in the quantum mechanical theory is

$$u = \frac{1}{2}\,\langle \mathrm{Tr}\,\phi^2 \rangle$$

It is easy to verify that they agree classically ($u \to \infty$), but $u = \bar{u} + \mathcal{O}(\Lambda^2)$.

The dynamically generated scale Λ *triggers* the quantum corrections. If we take $\Lambda \to 0$, any a will be larger which is a signal of semiclassical behavior.

In other words, $u = a^2 + \mathcal{O}(\Lambda^2)$. Obtaining the exact relation $u(a)$ is the aim of our lectures. In the region $u \to \infty$, due to asymptotic freedom,

$$a_D = \mathcal{F}'(a) = \frac{i}{\pi}\, a \left[\log\left(\frac{a^2}{\Lambda^2}\right) + 1\right] \simeq \frac{i}{\pi}\sqrt{u}\left[\log\left(\frac{u}{\Lambda^2}\right) + 1\right]$$