Lecture 9: The Seiberg-Witten solution

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The duality group

The duality group is $S\ell(2,\mathbb{Z})$. $\mathbb{M} \in S\ell(2,\mathbb{Z})$ reads

$$\mathbb{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with} \quad a,b,c,d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1$$

The central charge of a dyon reads

$$Z(n,m) = n\,a + m\,\tau_0\,a$$

The right quantum mechanical formula should be duality invariant

$$Z(n,m) = n\,a + m\,a_D = (m \quad n) \begin{pmatrix} a_D \\ a \end{pmatrix}$$

A duality transformation on (a_D, a) implies an analogous transformation of the charge vector (m, n).

The dual quantum numbers, $(\tilde{m}, \tilde{n})$, are still integers.

Browsing the moduli space

The classical moduli space, $\mathcal{M}_0$, has a faithful complex coordinate,

$$\bar{u} = \frac{1}{2}\,\mathrm{Tr}\,\langle\phi\rangle^2 = a^2$$

Now, the gauge invariant object relevant in the quantum mechanical theory is

$$u = \frac{1}{2}\,\langle \mathrm{Tr}\,\phi^2\rangle$$

It is easy to verify that they agree classically ($u \to \infty$), but $u = \bar{u} + \mathcal{O}(\Lambda^2)$.

Λ *triggers* the quantum corrections. If we take $\Lambda \to 0$, any a will be larger which is a signal of semiclassical behavior.

In other words, $u = a^2 + \mathcal{O}(\Lambda^2)$. Obtaining $u(a)$ is the goal. In the region $u \to \infty$, due to asymptotic freedom,

$$a_D = \mathcal{F}'(a) = \frac{i}{\pi}\,a\left[\log\left(\frac{a^2}{\Lambda^2}\right) + 1\right] \simeq \frac{i}{\pi}\sqrt{u}\left[\log\left(\frac{u}{\Lambda^2}\right) + 1\right]$$

Singularities in the quantum moduli space

Now, take u encircling clockwise the point at ∞ on the Riemann sphere,

$$a_D \to \frac{i}{\pi}\sqrt{e^{2\pi i}u}\left[\log\left(\frac{e^{2\pi i}u}{\Lambda^2}\right)+1\right] = -a_D + 2a \qquad a \to \sqrt{e^{2\pi i}u} = -a$$

(notice that a^2 is unaffected). In other words, when $u \to \infty$,

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \mathbb{M}_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad \text{where} \quad \mathbb{M}_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

We have pinpointed a singularity (a *branch point*) of $\mathcal{M}_\Lambda$ at $u = \infty$.

How many singularities are there in $\mathcal{M}_\Lambda$?

Since a branch point has to start and end somewhere, there must be at least one. Is it the singularity classically located at the origin of $\mathcal{M}_0$? Recall:

$U(1)_R \to \mathbb{Z}_8$ acts as $\phi \to e^{2i\alpha}\phi$ with $\alpha = 2\pi\frac{k}{8}$. Then, for odd k, $u \to -u$.

Three singularities for Muster Mark

Thus, singularities in $\mathcal{M}_\Lambda$ come in pairs, except at the origin. That is precisely a suspicious point where, at the classical level, there was a singularity.

Hence, if there are two singularities one should be at $u = 0$. But, by contour deformation, $\mathbb{M}_0 = \mathbb{M}_\infty$. Then a^2 would be a global coordinate on $\mathcal{M}_\Lambda$, and this is not possible. Two singularities are not enough.

If there are three singularities, we know that a couple of them will be placed at $u = \pm u_0$ for some $u_0 \neq 0$.

But what about the former classical singularity at the origin? Does it imply that there should be at least four singularities? NO.

Monodromy arguments plus number theory rule out this possibility: three!

The singularity of $\mathcal{M}_0$ disappears at quantum level! Clasically, $u = 0$ means no Higgs mechanism, thus, *Wilsonian effective action should break down*.

But classical reasoning is not valid when $a \to 0$!

The point $a = 0$ does not belong to $\mathcal{M}_\Lambda$.

The nature of the singularities

We might naively conjecture that there are still gauge bosons that become massless at $u = \pm u_0$. However, massless gauge bosons would imply an asymptotically conformally invariant theory in the IR.

Conformal invariance, in turn, implies $u = \frac{1}{2} \langle \mathrm{Tr}\, \phi^2 \rangle = 0$.

Singularities at $u = \pm u_0$ do not correspond to massless gauge bosons.

There are no other elementary $\mathcal{N} = 2$ multiplets in our theory. There should be collective excitations (magnetic monopoles, dyons) becoming massless at these singularities.

If there is a magnetic monopole, its (BPS) mass is $M = |a_D|$. Thus, this would imply that $a_D(u_0) = 0$. They are described by $\mathcal{N} = 2$ hypermultiplets that couple to Φ_D and W_D^α just as *'electrons'* would couple locally to Φ and W_α.

The Wilsonian theory in the vicinity of $u = u_0$ would be nothing but $\mathcal{N} = 2$ SQED with light *electrons* and a subscript D everywhere: an $\mathcal{N} = 2$ SQMD!

Weakly coupled massless monopoles

$\mathcal{N} = 2$ SQMD is not asymptotically free: it has a positive β function

$$\frac{d g_D}{d \log \mu} = \frac{{g_D}^3}{8\pi}$$

But the scale μ is provided by a_D and $\tau(a_D) = 4\pi i / {g_D}^2$ (θ_D vanishes in an Abelian theory). Thus, for $u \simeq u_0$ or $a_D \simeq 0$,

$$\frac{d\tau_D}{d \log a_D} = -\frac{i}{\pi} \qquad \Rightarrow \qquad \tau_D(a_D) = -\frac{i}{\pi} \log a_D$$

Notice that $g_D \simeq 0$ when $a_D \simeq 0$. Recalling that $\tau_D(a_D) = -a'(a_D)$,

$$a(a_D) \simeq a_0 + \frac{i}{\pi} a_D \log a_D$$

a_D should be a good coordinate around u_0, hence depend linearly on u,

$$a_D(u) \simeq c_0(u - u_0) \qquad a(u) \simeq a_0 + \frac{i}{\pi} c_0(u - u_0) \log(u - u_0)$$

There is a monodromy induced when $u - u_0 \to e^{2\pi i}(u - u_0)$.

Weakly coupled massless dyons

The monodromy can be easily computed

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \mathbb{M}_{u_0} \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad \text{where} \quad \mathbb{M}_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

(notice that a_D is unaffected, as a^2 around ∞). To obtain the monodromy at $-u_0$ it is sufficient to realize that

$$\mathbb{M}_\infty = \mathbb{M}_{u_0}\, \mathbb{M}_{-u_0} \qquad \Rightarrow \qquad \mathbb{M}_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

What is the massless state responsible of the singularity at $-u_0$? Recall the BPS mass spectrum

$$Z(n, m) = n\, a + m\, a_D = (m \quad n) \begin{pmatrix} a_D \\ a \end{pmatrix}$$

The monodromy transformation can be interpreted as acting on the magnetic and electric quantum numbers. The state of vanishing mass should be monodromy invariant, hence a left eigenvector of $\mathbb{M}_{-u_0}$ with unit eigenvalue.

Weakly coupled massless dyons

This is the case for $\mathbb{M}_{u_0}$ and the magnetic monopole. From $\mathbb{M}_{-u_0}$ we can see that the massless state is a $(1, -1)$ dyon. We can ask about the solution to

$$(m \quad n)\, \mathbb{M}(m,n) = (m \quad n)$$

which will give the monodromy matrix that should appear for a singularity due to a massless dyon with charges (m, n). One readily finds

$$\mathbb{M}(m,n) = \begin{pmatrix} 1+2mn & 2n^2 \\ -2m^2 & 1-2mn \end{pmatrix}$$

It is instructive to see that $\mathbb{M}_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$ does not have this form: it does not correspond to a hypermultiplet becoming massless.

If there were p strong coupling singularities, we should have

$$\mathbb{M}_\infty = \mathbb{M}_{u_1} \cdots \mathbb{M}_{u_p} \qquad \text{with} \qquad \mathbb{M}_{u_i} = \mathbb{M}(m_i, n_i)$$

There seems to be no solution to this equation for $p > 2$!

The Seiberg-Witten solution

After this long journey, we have arrived at the following *state of the art*:

- We know the form of the prepotential:

$$\mathcal{F}(a) = \frac{1}{2}\,\tau_0\, a^2 + \frac{i}{2\pi}\, a^2 \,\log\left(\frac{a^2}{\Lambda^2}\right) + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\mathcal{F}_k(a)\,\Lambda^{4\,k}$$

- We know that the moduli space, $\mathcal{M}_\Lambda$, is a complex u-plane with three singularities, whose meaning is by now clear.
- We know which are the good local coordinates for the patches including each singularity.
- We know the BPS spectrum of the theory, $Z(n,m) = n\,a + m\,a_D$.
- We know that the gauge coupling reads, in the semiclassical patch:

$$\tau(a^2) = \frac{i}{\pi}\left[3 + \log\left(\frac{a^2}{\Lambda^2}\right)\right] + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\mathcal{F}_k{}''(a)\,\Lambda^{4\,k}$$

  and, when going to the remaining patches, it should be transformed as a *modular parameter* of $S\ell(2,\mathbb{Z})$. Its imaginary part is positive.

The Seiberg-Witten solution

Taking into account all of the above, Nathan Seiberg and Edward Witten gave, in 1994, the exact answer:

$\star$ The gauge coupling, $\tau(a^2)$, is the period matrix of an elliptic curve.

$\star$ The elliptic curve is constructed in terms of $P_2(\lambda) = \lambda^2 - u$, and Λ,

$$y^2 = P_2^2(\lambda) - 4\Lambda^4 = (\lambda^2 - u)^2 - 4\Lambda^4 = (\lambda^2 - u - 2\Lambda^2)(\lambda^2 - u + 2\Lambda^2)$$

That is, $y^2 = y_+ \, y_-$,

$$y = (\lambda - e_1^+(u,\Lambda))(\lambda - e_2^+(u,\Lambda))(\lambda - e_1^-(u,\Lambda))(\lambda - e_2^-(u,\Lambda))$$

with four branch points $e_1^\pm = \sqrt{u \pm 2\Lambda^2}$, $e_2^\pm = -\sqrt{u \pm 2\Lambda^2}$.

$\star$ The curve degenerates whenever two branch points meet,

$$\Delta_\Lambda = \Delta_\Lambda^+ \, \Delta_\Lambda^- = \prod_{i<j} (e_i^+ - e_j^+)^2 \, (e_i^- - e_j^-)^2 = (2\Lambda)^8 (u + 2\Lambda^2)(u - 2\Lambda^2)$$

The Seiberg-Witten solution

Now, recall that $\Delta_0(\bar{u}) = 4\bar{u}$ led to the classical singular locus $\Sigma_0 = \bar{u}$.

Notice that $\Delta_\Lambda^\pm(u,\Lambda) \propto \Delta_0(u \pm 2\Lambda^2)$. The singular locus, hence, is

$$\Sigma_\Lambda(u,\Lambda) = \Sigma_0(u + 2\Lambda^2) \cup \Sigma_0(u - 2\Lambda^2)$$

Two singularities at $u = \pm u_0 = \pm 2\Lambda^2$, consistent with the classical limit.

$\star$ There is a special meromorphic differential, $d\lambda_{SW}$,

$$d\lambda_{SW} = d\lambda_{SW}(u,\Lambda) := \frac{\lambda\, P_2'(\lambda)}{y}\, d\lambda = \frac{\lambda\, P_2'(\lambda)}{\sqrt{P_2^2(\lambda) - 4\Lambda^4}}\, d\lambda$$

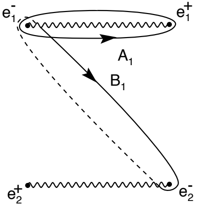
that leads us to the quantities $a(u)$ and $a_D(u)$ through its periods

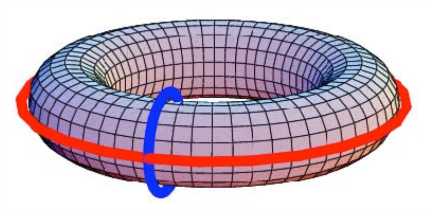
$$a(u) = \oint_A d\lambda_{SW} \qquad a_D(u) = \oint_B d\lambda_{SW} := \mathcal{F}'(a)$$

This provides an implicit expression for the exact Wilsonian $\mathcal{F}(a)$.

The BPS spectrum

The BPS spectrum is obtained by integrating $d\lambda_{SW}$ along non-trivial cycles of the torus, $\nu_{(n,m)} = n \cdot A + m \cdot B$,





$$M(n,m) = \sqrt{2}\ |Z(n,m)| = \sqrt{2}\ |n \cdot a + m \cdot a_D| = \sqrt{2} \left| \oint_\nu d\lambda_{SW} \right|$$

The symplectic group $S\ell(2,\mathbb{Z})$ acts on the homology basis as it does on the periods (a, a_D): duality group translates into the modular group of the torus.

Appart from the mass, what remains invariant is the *intersection number* of two BPS states, $\nu_{(n,m)} \cap \nu'_{(n',m')} = n \cdot m' - n' \cdot m$, which is an integer.

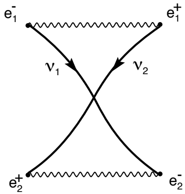
The BPS spectrum

This is the standard Dirac-Schwinger-Zwanzinger quantization condition for the possible electric and magnetic charges of dyons.

Two dyons are mutually local if they don't intersect $\Rightarrow$ there is a symplectic basis such that both states look purely electrically (or magnetically) charged,

$$\nu_{(n,0)} \cap \nu'_{(n',0)} = 0$$

At the singularities, $u = \pm 2\Lambda^2$, two of the four branch points, $e_1^\pm = \sqrt{u \pm 2\Lambda^2}$ and $e_2^\pm = -\sqrt{u \pm 2\Lambda^2}$, degenerate,



Massless BPS states at the singularities

This corresponds to the ν_1-cycle or the ν_2-cycle shrinking to zero size.

The BPS spectrum

These are the vanishing cycles. Expanding, $\nu_{(n,m)} = n \cdot A + m \cdot B$, allows us to read the quantum numbers of the BPS states becoming massless at $u = \pm u_0$.

The points of the singular locus can be understood as follows:

- $u = 2\Lambda^2$, $e_1^- = e_2^-$. The B_1-cycle shrinks to zero. This means $a_D = 0$. This corresponds to a magnetic monopole becoming massless.
- $u = -2\Lambda^2$, $e_1^+ = e_2^+$. The $(-2A_1 + B_2)$-cycle, shrinks, thus $(a_D - 2\,a\,) = 0$. It corresponds to a dyon.

Notice that this scheme also reproduces the classical BPS spectrum. Indeed, when $\Lambda \to 0$, the curve $y^2 = P_2^2(\lambda) - 4\Lambda^4 \;\to\; y = P_2(\lambda) = (\lambda - e_1)(\lambda - e_2)$,

$$d\lambda_{SW} \to \frac{\lambda\, P_2'(\lambda)}{P_2(\lambda)}\, d\lambda = \sum_{i=1}^{2} \frac{\lambda}{\lambda - e_i}\, d\lambda \;\Rightarrow\; z_a = \oint_\nu d\lambda_{SW} = e_1 - e_2 = 2a$$

where ν is an eight-shaped cycle surrounding both roots and $M = \sqrt{2}\;|z_a|$.

The Seiberg-Witten solution

With the help Seiberg-Witten 1-form, $d\lambda_{SW}$, we can compute the cycles

$$a_D(u) = \oint_{B_1} d\lambda_{SW} = \frac{i}{2}\Lambda \left(\frac{u^2}{4\Lambda^4} - 1\right) {}_2F_1\left(\frac{3}{4}, \frac{3}{4}, 2; 1 - \frac{u^2}{4\Lambda^4}\right)$$

$$a(u) = \oint_{A_1} d\lambda_{SW} = \frac{2}{1+i}\Lambda \left(1 - \frac{u^2}{4\Lambda^4}\right)^{1/4} {}_2F_1\left(-\frac{1}{4}, \frac{3}{4}, 1; \frac{1}{1 - \frac{u^2}{4\Lambda^4}}\right)$$

where ${}_2F_1$ is the standard hypergeometric function. Inverting $a(u)$ is not an easy job but certainly can be done as a series for large a/Λ yielding

$$u = a^2 + \frac{1}{2}\left(\frac{\Lambda^4}{a^2}\right) + \frac{5}{32}\left(\frac{\Lambda^8}{a^6}\right) + \cdots$$

After inserting this into $a_D(u)$ one obtains, by integration with respect to a,

$$\mathcal{F}(a) = \frac{i}{2\pi} a^2 \left[\log\left(\frac{a^2}{\Lambda^2}\right) + 2\log 2 - 3 - \frac{1}{4}\frac{\Lambda^4}{a^4} - \frac{5}{128}\frac{\Lambda^8}{a^8} - \frac{3}{128}\frac{\Lambda^{12}}{a^{12}} - \cdots\right]$$

Physics in the vicinity of a singularity

How to proceed close to the singularity $a_D = 0$?

There is a BPS massless monopole multiplet, $M, \widetilde{M}$, at that point that has been wrongly integrated out.

We have to incorporate it, once again, into our Wilsonian description,

$$\mathcal{L} = \int d^4\theta \left(M^\dagger e^{-2V_D} M + \widetilde{M} e^{2V_D} \widetilde{M}^\dagger \right) + \sqrt{2} \int d^2\theta \; \widetilde{M} \, \Phi_D \, M + \mathcal{L}_{\mathrm{QMD}}$$

where we used the fact that the monopole is an $\mathcal{N} = 2$ hypermultiplet.

Let us analyze the effect of turning on a bare mass m for Φ, by adding

$$W(\Phi) = m \, \mathrm{Tr} \, \Phi^2$$

that naturally undertakes the supersymmetry breaking, $\mathcal{N} = 2 \to \mathcal{N} = 1$.

$\mathrm{Tr}\,\Phi^2$ is a chiral superfield U; scalar component is $u = \langle \mathrm{Tr}\,\phi^2 \rangle$. If $m^2 \ll u$, the Wilsonian effective action simply receives a contribution $W(u) = m\,u$.

Monopole condensation and confinement

This term lifts the vacuum degeneracy. In order to have mass gap, extra light charged fields should condense, and this happens at the singularities.

Near the point at which there is a massless monopole, $a_D \simeq 0$, we go to a dual description of the theory and introduce the monopole superfield,

$$W_{\text{eff}} = \sqrt{2}\, a_D\, M\, \widetilde{M} + m\, u(a_D)$$

We shall find solutions of $d\,W_{\text{eff}} = 0$:

- If $m = 0$, then $M = \widetilde{M} = 0$, and a_D is arbitrary.
- If $m \neq 0$, then $a_D\, M = a_D\, \widetilde{M} = 0$, and $\sqrt{2}\, M\, \widetilde{M} + m\, u'(a_D) = 0$. Hence, if $u'(a_D) \neq 0$, $a_D = 0$ and monopoles condense,

$$M = \widetilde{M} = \left(-\frac{m}{\sqrt{2}}\, u'(0) \right)^{1/2}$$

confinement being triggered since M is charged: its *vacuum expectation value* generates a mass for the gauge field.

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory

As mentioned above, $\mathcal{N} = 4$ is the maximum amount of supersymmetry for a QFT without gravity in four dimensions.

Similar to what we saw earlier, the $\mathcal{N} = 4$ vector multiplet can be written in terms of an $\mathcal{N} = 2$ chiral multiplet and an $\mathcal{N} = 2$ hypermultiplet; thus, it has 4 Weyl fermions and 3 complex scalars.

The Lagrangian is unique and given by

$$\begin{aligned}\mathcal{L} \quad = \quad \mathrm{Tr}\Big[& -\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i \\ & + \sum_{a,b,i} g\, C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g\, \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \Big]\end{aligned}$$

The theory is conformal

$$\beta(g) = \mu \frac{dg}{d\mu} = -\frac{g^3}{16\pi} \left(\frac{11}{3} - \frac{2}{3} \times 4 - \frac{1}{3} \times 3 \right) C_2(G) = 0$$

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory

The only object you can play with is the coupling constant parameter τ,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

The theory is not only renormalizable but finite: perturbative quantization does not lead to UV divergences in the correlation functions. Instanton corrections also lead to finite contributions.

- ⋆ So-called S-duality: $S\ell(2,\mathbb{Z})$ is a symmetry at the quantum level

$$\tau \to \frac{a\,\tau + b}{c\,\tau + d} \qquad a, b, c, d \in \mathbb{Z}$$

In particular, $\tau \to -1/\tau$ (strong/weak coupling duality).

- ⋆ The Lagrangian is scale invariant (all terms are of dimension 4). Now,

$$\text{Scale invariance} + \text{Poincare invariance} + \mathcal{N} = 4 \ \text{SUSY} = SU(2,2|4)$$

The superconformal group $SU(2,2|4)$ is a quantum mechanical symmetry.