1 Spinors

We will use the metric convention:

$$g^{\mu\nu} = (1, -1, -1, -1) \ . \tag{1.1}$$

We will also use the Weyl representation of the Dirac spinors:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ , \tag{1.2}$$

where σ^μ and $\bar{\sigma}^\mu$ are the following 2×2 matrices:

$$\sigma^\mu = (\sigma^0, \sigma^i) \ , \qquad \bar{\sigma}^\mu = (\sigma^0, -\sigma^i) \ , \tag{1.3}$$

with σ^i being the Pauli matrices that satisfy the relation:

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \ . \tag{1.4}$$

The matrices σ^μ and $\bar{\sigma}^\nu$ satisfy the relations:

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2g^{\mu\nu} \ , \qquad \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu} \ . \tag{1.5}$$

Let us define:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{1.6}$$

In this representation the upper [bottom] two components have left [right] chirality:

$$\Psi = \Psi_L + \Psi_R \ , \tag{1.7}$$

with

$$\Psi_L = \left(\frac{1 - \gamma_5}{2}\right) \Psi \ , \qquad \Psi_R = \left(\frac{1 + \gamma_5}{2}\right) \Psi \ . \tag{1.8}$$

Let us represent Ψ in terms of two-component spinors as:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \ . \tag{1.9}$$

Then, the left and right chirality spinor fields are:

$$\Psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \ , \qquad \Psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \ . \tag{1.10}$$

The Dirac conjugate is defined as:

$$\bar{\Psi} \equiv \Psi^\dagger \gamma^0 \ . \tag{1.11}$$

With our conventions we have:

$$\bar{\Psi} \;=\; \left(\, \bar{\chi}^{\dot{\alpha}\,\dagger} \,,\, \psi_{\alpha}^{\dagger} \,\right) \tag{1.12}$$

Let us define

$$\bar{\psi}_{\dot{\alpha}} = \left[\psi_{\alpha}\right]^{\dagger} \;, \qquad \chi^{\alpha} = \left[\bar{\chi}^{\dot{\alpha}}\right]^{\dagger} \;. \tag{1.13}$$

Then

$$\bar{\Psi} \;=\; \left(\, \chi^{\alpha} \,,\, \bar{\psi}_{\dot{\alpha}} \,\right) \;. \tag{1.14}$$

Raising and lowering undotted and dotted indices can be done with the matrices

$$\epsilon_{\alpha\beta} \;=\; \epsilon_{\dot{\alpha}\dot{\beta}} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;=\; i\sigma^2 \;, \qquad \epsilon^{\alpha\beta} \;=\; \epsilon^{\dot{\alpha}\dot{\beta}} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;=\; -i\sigma^2 \;. \tag{1.15}$$

They satisfy:

$$\epsilon^{\gamma\alpha}\,\epsilon_{\alpha\lambda} \;=\; \delta^{\gamma}_{\lambda} \;, \qquad\qquad \epsilon^{\dot{\gamma}\dot{\alpha}}\,\epsilon_{\dot{\alpha}\dot{\lambda}} \;=\; \delta^{\dot{\gamma}}_{\dot{\lambda}} \;. \tag{1.16}$$

One has:

$$\begin{aligned} \chi_{\alpha} \;&=\; \epsilon_{\alpha\beta}\,\chi^{\beta} \;, \qquad \chi^{\alpha} \;=\; \epsilon^{\alpha\beta}\,\chi_{\beta} \;, \\ \bar{\psi}^{\dot{\alpha}} \;&=\; \epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\psi}_{\dot{\beta}} \;, \qquad \bar{\psi}_{\dot{\alpha}} \;=\; \epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\psi}^{\dot{\beta}} \;. \end{aligned} \tag{1.17}$$

So, for example:

$$\chi^{1} \;=\; -\chi_2 \;, \qquad \chi^{2} \;=\; \chi_1 \;. \tag{1.18}$$

1.1 Charge conjugation

In Dirac theory the charge conjugated Dirac spinor is given by:

$$\Psi^c = C\bar{\Psi}^T \;, \tag{1.19}$$

where C is a matrix that must satisfy:

$$C\gamma_{\mu}^{T}\,C^{-1} \;=\; -\gamma_{\mu} \;. \tag{1.20}$$

We will take the matric C as:

$$C = -i\gamma^0\gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} \;. \tag{1.21}$$

In terms of two-component spinors the charge conjugate of Ψ is:

$$\Psi^c = \begin{pmatrix} \chi_{\alpha} \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \;. \tag{1.22}$$

A Majorana spinor satisfies the property of self-conjugation:

$$\Psi_M = \Psi_M^c \;. \tag{1.23}$$

One can prove that this is equivalent to be of the form:

$$\Psi_M = \begin{pmatrix} \chi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \chi \\ -i\sigma_2\,\chi^* \end{pmatrix} \;. \tag{1.24}$$

1.2 Lorentz invariance

A Dirac spinor is transformed under a Lorentz transformation as:

$$\Psi \to S\,\Psi\ , \tag{1.25}$$

where S is the following matrix:

$$S = e^{-\frac{i}{4}\omega_{\mu\nu}\Sigma^{\mu\nu}}\ , \tag{1.26}$$

and $\Sigma^{\mu\nu}$ is:

$$\frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} i\sigma^{\mu\nu} & 0 \\ 0 & i\bar{\sigma}^{\mu\nu} \end{pmatrix}\ , \tag{1.27}$$

with $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ being given by:

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} \equiv \frac{1}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha^{\ \beta}\ , \qquad (\bar{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \equiv \frac{1}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot\alpha}_{\ \dot\beta}\ . \tag{1.28}$$

These expressions are consistent with the following index structures:

$$(\sigma^\mu)_{\alpha\dot\alpha}\ , \qquad\qquad (\bar{\sigma}^\mu)^{\dot\alpha\alpha}\ . \tag{1.29}$$

This index assignment is also consistent with the property:

$$(\bar{\sigma}^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}\ . \tag{1.30}$$

We now define M as:

$$M = e^{\frac{1}{2}\,\omega_{\mu\nu}\,\sigma^{\mu\nu}}\ . \tag{1.31}$$

It satisfies

$$\left(M^\dagger\right)^{-1} = e^{\frac{1}{2}\,\omega_{\mu\nu}\,\bar{\sigma}^{\mu\nu}}\ . \tag{1.32}$$

S can be written in terms of M as:

$$S = \begin{pmatrix} M & 0 \\ 0 & \left(M^\dagger\right)^{-1} \end{pmatrix}\ . \tag{1.33}$$

This implies that the different two-component spinors transform as:

$$\begin{aligned} \psi_\alpha &\to [M\psi]_\alpha\ , & \psi^\alpha &\to [\psi M^{-1}]^\alpha\ , \\ \bar{\chi}^{\dot\alpha} &\to \left[\left(M^\dagger\right)^{-1}\bar{\chi}\right]^{\dot\alpha}\ , & \bar{\chi}_{\dot\alpha} &\to \left[\bar{\chi}\,M^\dagger\right]_{\dot\alpha}\ . \end{aligned} \tag{1.34}$$

It follows that $\psi^\alpha\chi_\alpha$ and $\bar{\psi}_{\dot\alpha}\bar{\chi}^{\dot\alpha}$ are scalars. We shall denote:

$$\chi\,\psi \equiv \chi^\alpha\,\psi_\alpha\ , \qquad \bar{\chi}\,\bar{\psi} \equiv \bar{\chi}_{\dot\alpha}\,\bar{\psi}^{\dot\alpha}\ . \tag{1.35}$$

Notice that :

$$\chi^\alpha\,\psi_\alpha = -\chi_\alpha\,\psi^\alpha\ , \qquad \bar{\chi}_{\dot\alpha}\,\bar{\psi}^{\dot\alpha} = -\bar{\chi}^{\dot\alpha}\,\bar{\psi}_{\dot\alpha}\ . \tag{1.36}$$

For anticommuting spinors one has:

$$\chi\,\psi \;=\; \psi\,\chi\;, \qquad \bar{\chi}\,\bar{\psi} \;=\; \bar{\psi}\,\bar{\chi}\;. \tag{1.37}$$

The 4×4 matrix S that implements Lorentz transformations on the four-component Dirac spinors satisfies:

$$S^{-1}\,\gamma^\mu\,S \;=\; \Lambda^\mu_{\;\nu}\,\gamma^\nu\;. \tag{1.38}$$

It follows that the 2×2 matrix M satisfies:

$$M^{-1}\,\sigma^\mu\,\left(M^\dagger\right)^{-1} \;=\; \Lambda^\mu{}_\nu\,\sigma^\nu\;, \qquad\qquad M^\dagger\,\bar{\sigma}^\mu\;M \;=\; \Lambda^\mu{}_\nu\,\bar{\sigma}^\nu\;. \tag{1.39}$$

Then:

$$\psi^\alpha\left(\sigma^\mu\right)_{\alpha\,\dot{\alpha}}\,\bar{\chi}^{\dot{\alpha}} \;=\; \psi\,\sigma^\mu\bar{\chi}\quad \text{is a vector} \tag{1.40}$$

$$\bar{\chi}_{\dot{\alpha}}\left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\,\alpha}\,\psi_\alpha \;=\; \bar{\chi}\,\bar{\sigma}^\mu\bar{\psi}\quad \text{is a vector} \tag{1.41}$$

1.3 Spinor identities

When computing the complex conjugate of fermionic bilinears the identities that follow are very useful:

$$\left(\psi\,\chi\right)^\dagger \;=\; \bar{\chi}\,\bar{\psi}\;, \qquad\qquad \left(\bar{\psi}\,\bar{\chi}\right)^\dagger \;=\; \chi\,\psi\;,$$

$$(\chi\sigma^\mu\,\bar{\psi})^\dagger \;=\; \psi\,\sigma^\mu\,\bar{\chi}\;, \qquad\qquad (\bar{\psi}\,\sigma^\mu\,\chi)^\dagger \;=\; \bar{\chi}\,\bar{\sigma}^\mu\,\psi\;. \tag{1.42}$$

One can also exchange the order of spinors in a blilinear by means of the identities:

$$\psi\,\sigma^\mu\,\bar{\chi} \;=\; -\bar{\chi}\,\bar{\sigma}^\mu\,\psi\;,$$

$$\chi\,\sigma^{\mu\nu}\,\psi \;=\; -\psi\,\sigma^{\mu\nu}\,\chi\;, \qquad\qquad \bar{\chi}\,\bar{\sigma}^{\mu\nu}\,\bar{\psi} \;=\; -\bar{\psi}\,\bar{\sigma}^{\mu\nu}\,\bar{\chi}\;. \tag{1.43}$$

Working in superspace with coordinates θ and $\bar{\theta}$ one frequently uses the Fierzing identities:

$$\theta^\alpha\,\theta^\beta \;=\; -\frac{1}{2}\,\epsilon^{\alpha\beta}\;\theta\theta\;, \qquad\qquad \theta_\alpha\,\theta_\beta \;=\; \frac{1}{2}\,\epsilon_{\alpha\beta}\;\theta\theta\;,$$

$$\bar{\theta}^{\dot{\alpha}}\,\bar{\theta}^{\dot{\beta}} \;=\; \frac{1}{2}\,\epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\theta}\bar{\theta}\;, \qquad\qquad \bar{\theta}_{\dot{\alpha}}\,\bar{\theta}_{\dot{\beta}} \;=\; -\frac{1}{2}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\theta}\bar{\theta}\;,$$

$$\theta\psi\;\theta\chi \;=\; -\frac{1}{2}\,\theta\theta\;\psi\chi\;, \qquad \bar{\theta}\bar{\psi}\;\bar{\theta}\bar{\chi} \;=\; -\frac{1}{2}\,\bar{\theta}\bar{\theta}\;\bar{\psi}\bar{\chi}\;. \tag{1.44}$$

From the completeness property of the σ-matrices:

$$\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\,\left(\bar{\sigma}_\mu\right)^{\dot{\beta}\beta} \;=\; 2\,\delta_\alpha^\beta\,\delta_{\dot{\alpha}}^{\dot{\beta}}\;, \tag{1.45}$$

we get:

$$\theta^{\alpha}\,\bar{\theta}^{\dot{\alpha}} \;=\; \frac{1}{2}\,\big(\,\bar{\sigma}^{\mu}\,\big)^{\dot{\alpha}\alpha}\;\theta\sigma_{\mu}\,\bar{\theta}\;. \tag{1.46}$$

Also:

$$\theta\sigma^{\mu}\,\bar{\theta}\;\theta\sigma^{\nu}\,\bar{\theta} \;=\; \frac{1}{2}\,\theta\theta\;\bar{\theta}\bar{\theta}\,g^{\mu\nu}\;. \tag{1.47}$$

Another useful properties are:

$$\epsilon^{\alpha\beta}\,\frac{\partial}{\partial\theta^{\beta}} \;=\; -\frac{\partial}{\partial\,\theta_{\alpha}}\;, \qquad\qquad \epsilon^{\dot{\alpha}\dot{\beta}}\,\frac{\partial}{\partial\theta^{\dot{\beta}}} \;=\; -\frac{\partial}{\partial\,\theta_{\dot{\alpha}}}\;. \tag{1.48}$$

1.4 Properties of Pauli matrices

The traces of σ matrices are given by:

$$\begin{aligned}
\mathrm{Tr}\Big[\,\sigma^{\mu}\,\bar{\sigma}^{\nu}\,\Big] &\;=\; 2g^{\mu\nu}\;,\\
\mathrm{Tr}\Big[\,\sigma^{\mu\nu}\,\sigma^{\rho\sigma}\,\Big] &\;=\; \frac{1}{2}\Big(\,g^{\mu\sigma}g^{\nu\rho}\;-\;g^{\mu\rho}g^{\nu\sigma}\;+\;i\epsilon^{\mu\nu\rho\sigma}\,\Big)\;,\\
\mathrm{Tr}\Big[\,\bar{\sigma}^{\mu\nu}\,\bar{\sigma}^{\rho\sigma}\,\Big] &\;=\; \frac{1}{2}\Big(\,g^{\mu\sigma}g^{\nu\rho}\;-\;g^{\mu\rho}g^{\nu\sigma}\;+\;i\epsilon^{\mu\nu\rho\sigma}\,\Big)\;.
\end{aligned} \tag{1.49}$$

The hermitian conjugate of the σ matrices are given by:

$$\sigma^{\mu\,\dagger} \;=\; \sigma^{\mu}\;, \qquad \bar{\sigma}^{\mu\,\dagger} \;=\; \bar{\sigma}^{\mu}\;, \qquad \sigma^{\mu\nu\,\dagger} \;=\; -\bar{\sigma}^{\mu\nu}\;. \tag{1.50}$$

The transposes of the σ's are obtained by conjugating with σ^2:

$$\begin{aligned}
\sigma^{\mu\,T} &\;=\; \sigma^{2}\,\bar{\sigma}^{\mu}\,\sigma^{2}\;, & \bar{\sigma}^{\mu\,T} &\;=\; \sigma^{2}\,\sigma^{\mu}\,\sigma^{2}\;,\\
(\,\sigma^{\mu\nu}\,)^{T} &\;=\; -\sigma^{2}\,\sigma^{\mu\nu}\,\sigma^{2}\;, & (\,\bar{\sigma}^{\mu\nu}\,)^{T} &\;=\; -\sigma^{2}\,\bar{\sigma}^{\mu\nu}\,\sigma^{2}\;.
\end{aligned} \tag{1.51}$$