Notes on Supersymmetry

Universidad de Santiago de Compostela

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Capítulo 1

Non perturbative aspects of N=1 Supersymmetric Theory

1.1. $SU(N_c)$-QCD with N_f flavours.

Let us start by defining the class of microscopic models we will be dealing with in this course. They represent the minimal supersymmetric extension of QCD, and hence receive the generic name of SQCD.

1.1.1. Field Content

- *Chiral Matter.*

  A single flavour is composed of a pair of chiral superfields $(Q^i, \tilde{Q}_i),\ i = 1, ..., N_c$. With $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$

$$
\begin{aligned}
Q^i(x,\theta,\bar{\theta}) &= \phi^i(y) + \sqrt{2}\theta\psi^i(y) + \theta^2 F^i(y) \\
\tilde{Q}_i(x,\theta,\bar{\theta}) &= \tilde{\phi}_i(y) + \sqrt{2}\theta\tilde{\psi}_i(y) + \theta^2 \tilde{F}_i(y)
\end{aligned}
$$

  The component fields receive the generic names of *squark* (ϕ^i), and *quark* fields $(\psi^i_\alpha, \tilde{\psi}_{\alpha\, i})$.

- *Gauge Fields*

  In the Wess Zumino gauge $V = T_a W^a = V^\dagger$ can be expanded as

$$
V^a(x,\theta,\bar{\theta}) = \theta\sigma^\mu\bar{\theta}A^a_\mu(x) + i\theta^2(\bar{\theta}\bar{\lambda}^a) - i\bar{\theta}^2(\theta\lambda)^a + \frac{1}{2}\theta^2\bar{\theta}^2 D^2(x) \tag{1.1}
$$

  in terms of a gaugino $(\lambda^a_\alpha, \bar{\lambda}^{a\dot{\alpha}})$, and a gauge boson A^a_μ. The gauge invariant field strength

$$
W_\alpha(x,\theta,\bar{\theta}) = -\frac{1}{4}\bar{D}^2 e^{-2V} D_\alpha e^{2V} \tag{1.2}
$$

$$
= \left(-i\lambda^a(y) + \theta_\alpha D^a(y) - i(\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) - \theta^2\sigma^\mu_{\alpha\dot{\alpha}}(\mathcal{D}_\mu\bar{\lambda}^{\dot{\alpha}})^a(y)\right) T_a \tag{1.3}
$$

is a chiral superfield, *i.e.* $\bar{D}W = 0$. From here

$$\mathrm{Tr}W^2\big|_{\theta^2} = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + 2i\lambda^a\sigma^\mu D_\mu\bar{\lambda}_a + D^2 + \frac{i}{2}F^*_{\mu\nu}F^{\mu\nu} \tag{1.4}$$

hence

$$\begin{aligned} \frac{1}{4g^2}(\,\mathrm{Tr}W^2\big|_{\theta^2} + h.c.) &= -\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{g^2}\lambda\sigma^\mu D_\mu\bar{\lambda}_a + \frac{1}{2g^2}D^2 \\ -i\frac{\theta}{32\pi^2}(\,\mathrm{Tr}W^2\big|_{\theta^2} - h.c.) &= \frac{\theta}{32\pi^2}F^*_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{1.5}$$

with

$$\begin{aligned} F^*_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \\ \mathcal{D}_\mu\bar{\lambda}^{\dot{\alpha}} &= \partial_\mu\bar{\lambda}^{\dot{\alpha}} + [A_\mu, \bar{\lambda}^{\dot{\alpha}}] \end{aligned} \tag{1.6}$$

The gauge transformation of the different fields that enter is given in terms of a chiral superfield $\Lambda^i{}_j = \Lambda^a(T_a)^i{}_j$ and its antichiral hermitian conjugate $\Lambda^\dagger = \Lambda^{\dagger a}T_a^\dagger = \Lambda^{\dagger a}T_a$

- Q^i transforms in the N_c

$$Q^i \to (e^{-2i\Lambda})^i{}_j Q^j \ ; \ Q^\dagger_j = Q^\dagger_i(e^{2i\Lambda^\dagger})^i{}_j \tag{1.7}$$

- $\tilde{Q}_i$ transforms in the $\bar{N}_c$, or [1]

$$\tilde{Q}_i \to \tilde{Q}_i(e^{2i\Lambda})^i{}_j \ ; \ \tilde{Q}^{\dagger i} \to (e^{-2i\Lambda^\dagger})^i{}_j\tilde{Q}^{\dagger\, j} \tag{1.8}$$

- e^{2V} transforms in the adjoint $(N_c, \bar{N}_c)$ representation

$$e^{2V} \to e^{-2i\Lambda^\dagger}e^{2V}e^{2i\Lambda} \quad ; \quad e^{-2V} \to e^{-2i\Lambda}e^{-2V}e^{2i\Lambda^\dagger} \tag{1.9}$$

so that the combination $Q^\dagger e^{2V}Q + \tilde{Q}e^{-2V}\tilde{Q}^\dagger$ is gauge invariant. Also for the chiral field strength, using $[\bar{D}^{\dot{\alpha}}, e^{i\Lambda}] = [D_\alpha, e^{i\bar{\Lambda}}] = 0$ we find

$$\begin{aligned} W_\alpha = -\frac{1}{8}\bar{D}^2e^{-2V}D_\alpha e^{2V} \quad &\Longrightarrow \quad W_\alpha \to e^{-2i\Lambda}W_\alpha e^{2i\Lambda} \\ \bar{W}_{\dot{\alpha}} = -\frac{1}{8}D^2e^{2V}\bar{D}_{\dot{\alpha}}e^{-2V} \quad &\Longrightarrow \quad \bar{W}_{\dot{\alpha}} \to e^{-2i\Lambda^\dagger}W_\alpha e^{2i\Lambda^\dagger} \end{aligned} \tag{1.10}$$

[1] Remember that given a representation $D(g)D(g') = D(gg')$ of a group, automatically we have two new ones, $D(g)^*$ and $D(g)^{-1t}$ which in principle are inequivalent. The associated Lie algebra $D(g) = 1 + i\alpha_i\mathcal{D}(L^i) + ...$ has correspondingly inequivalent representations $-\mathcal{D}^t$, $D(g)^{-1t} = 1 - i\alpha_i\mathcal{D}^t(L^i) + ...$ and $D(g)^* \sim 1 - i\alpha_i\mathcal{D}^*(L^i) + ...$. With $\mathcal{D}(L^i)$ hermitian $\mathcal{D}(L^i)^t = \mathcal{D}(L^i)^*$, these are equivalent *only for real parameters* $\alpha_i \in$R. However, for general gauge transformations the parameter functions in (1.7) are chiral superfields $\Lambda \neq \Lambda^*$. Hence by $\bar{N}_c$ we must specify what we mean, and the choice is D^{-1t}, whence (1.8).

1.1.2. Action and symmetries

The action can be written in superspace notation as follows

$$\boxed{\begin{aligned}\mathcal{L} &= \mathrm{Im}\left(\frac{\tau}{8\pi}\mathrm{Tr}\int d^2\theta W^2\right) + \frac{1}{4}\int d^2\theta d^2\bar{\theta}\left(Q_f^\dagger e^{2V} Q^f + \tilde{Q}_f e^{-2V}\tilde{Q}^{\dagger f}\right) \\ &+ \left(\int d^2\theta\, \mathcal{W}(Q^f, \tilde{Q}_f) + h.c.\right)\end{aligned}} \tag{1.11}$$

where we have extended our chiral superfields to a set $(Q^{fi}, \tilde{Q}_{fi})$, $i = 1, ..., N_c; f = 1, ..., N_f$,

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{1.12}$$

hence the gauge-kinetic term reads as follows

$$\begin{aligned}\frac{\mathrm{Im}}{8\pi}\left(\tau\,\mathrm{Tr}\int d^2\theta W^2\right) &= \frac{1}{4g^2}\left(\int d\theta^2\,\mathrm{Tr}W^2 + h.c.\right) - i\frac{\theta}{32\pi^2}\left(\int d\theta^2\mathrm{Tr}W^2 - h.c.\right) \\ &= \mathrm{Tr}\left(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{g^2}\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2g^2}D^2 + \frac{\theta}{32\pi^2}\tilde{F}_{\mu\nu}F^{\mu\nu}\right)\end{aligned} \tag{1.13}$$

The superpotential contains typically mass as well as higher interaction terms

$$\mathcal{W}(Q, \tilde{Q}) = m_f^g Q^{fi}\tilde{Q}_{gi} + a_{fgh}\epsilon_{ijk}Q^{fi}Q^{gj}Q^{hk} + \tilde{a}_{fgh}\epsilon_{ijk}\tilde{Q}^{fi}\tilde{Q}^{gj}\tilde{Q}^{hk} + ... \tag{1.14}$$

where $a_{fgh}, \tilde{a}_{fgh}$ are $SU(N)$ invariant tensors proportional to ϵ_{fgh} [1] p.31. However these higher terms violate baryon number conservation.

1.1.2.1 Scalar Potential

$$\begin{aligned}V &= \sum_{a=1}^{N_c^2-1}|D^a|^2 + \sum_f(|F_{Q^f}|^2 + |F_{\tilde{Q}_f}|^2) \\ &= \frac{1}{2}\sum_{a=1}^{N_c^2-1}|\phi^{*f}T^a\phi_f - \tilde{\phi}_f T^a\tilde{\phi}^{*f}|^2 + \sum_{f,i}\left(\frac{\partial\mathcal{W}}{\partial\phi^{fi}}\frac{\partial\bar{\mathcal{W}}}{\partial\phi^\dagger_{fi}} + (\phi\leftrightarrow\tilde{\phi})\right)\end{aligned} \tag{1.15}$$

1.1.2.2 Global symmetries

In the absence of superpotential ($\mathcal{W} = 0$) $\rightarrow U(N_f)\times U(N_f)\times U(1)_R \sim SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)_R\times U(1)_A$.

	$SU(N_f)_L$	$SU(N_f)_R$	$U_B(1)$	$U_A(1)$	$U_R(1)$
V	0	0	0	0	0
Q^f	N_f	0	1	1	0
$\tilde{Q}_f$	0	$\bar{N}_f$	-1	1	0

$U(1)_R$ is a geometrical symmetry

$$\begin{aligned}(\theta,\bar\theta) &\to (e^{i\alpha}\theta, e^{-i\alpha}\bar\theta),\\ (d\theta,d\bar\theta) &\to (e^{-i\alpha}d\theta, e^{i\alpha}d\bar\theta)\end{aligned}$$

thus although the chiral superfields are neutral, the higher components inside are charged. In the presence of a superpotential of the form (1.14) and vanishing trilinear couplings, the global symmetry reduces to $SU(N)_V \times U(1)_B$ in the case of diagonal equal masses $m^{f_g} = m\delta^f{}_g$ and to $U(1)_B$ for generic $m^f{}_g$.

<u>1.1.2.3</u> *Theta term*

The gauge kinetic term (1.13) contains a θ-term. Remember that θ is a periodic variable. In other words $\theta \to \theta + 2\pi$ is a symmetry of the quantum theory. This is because spacetime integral of this term computes the"winding number" of the gauge field at infinity. Namely, there exist non-trivial gauge field configurations for which

$$\frac{\theta}{32\pi^2}\int d^4x\tilde{F}F = n\theta \tag{1.16}$$

with n an integer. Thus, although strictly speaking $\theta \to \theta + 2\pi$ is not a symmetry of S, it shifts trivialy the phase factor in the path integral.

1.2. SQCD Classical Moduli Space: $\mathcal{M}_0$

1.2.1. Classical Moduli Spaces

A common feature of supersymmetric models is the existence of a large amount of vacua, spanning continuous manifolds which generically receive the name of "moduli space". Moreover, as we shall see, quantum corrections do not generically lift this degeneracy of vacuum states. This is unlike non-supersymmetric theories, for which vacua are usually discrete sets of points. The vacuum condition is simply $V = 0$. The space $\mathcal{M}_0$ is in many cases not a manifold, but rather a set of manifold or "branches", often joined along subspaces of lower dimension.

As an example consider a model with 3 chiral multiplets X, Y and Z, and a superpotential $\mathcal{W} = \lambda XYZ$. The susy vacuum condition states that $xy = xz = yz = 0$ for the scalar component. So we find 3 branches, those are ($y = z = 0$ with x arbitrary) and cyclic permutations $x \to y \to z \to x$ of the same statement. The three branches join at a single point $x = y = z$ which is a singular point of $\mathcal{W}$ (all its gradients vanish).

1.2.2. $\mathcal{M}_0$(SQCD)

Let us start by examining this manifold in the case of the lagrangian (1.11) with a vanishing superpotential to start with. Later on we will consider the quantum modification of this lagrangian, and we will show that nonperturbative effects allow for the appearance of a nontrivial $\mathcal{W}$.

With $\mathcal{W} = 0$, the vacuum conditions can be read from setting $V = 0$ in (1.15), and therefore they also are called D-flatness conditions

$$\sum_a |D^a|^2 = \sum_a |\phi^{*f} T^a \phi_f - \tilde{\phi}_f T^a \tilde{\phi}^{*f}|^2 = 0 \tag{1.17}$$

The solution to these algebraic equations fall into three categories, whether $N_f < N_c$, $N_f = N_c$, $N_f = N_c + 1$ or $N_f > N_c + 1$. Let us examine them case by case.

- $\underline{N_f < N_c}$: Performing an $SU(N_f) \times SU(N_c)$ rotation we may bring ϕ^{fi} and $\tilde{\phi}_{fi}$ to the form

$$\phi^{fi} = a_f \delta^{fi} = \overbrace{\begin{pmatrix} a_1 & 0 & \cdots & 0 & \cdots \\ 0 & a_2 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{N_f} & \cdots \end{pmatrix}}^{N_c} \left.\vphantom{\begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix}}\right\} N_f \tag{1.18}$$

and the same for $\tilde{\phi}_{fi} = \tilde{a}_f \delta_{fi}$. Hence (1.17) yields

$$\sum_a \sum_{f=1}^{N_f < N_c} \left(|a_f|^2 - |\tilde{a}_f|^2\right) (T^a)^f{}_f = 0 \tag{1.19}$$

This has the generic solution

$$\boxed{|a_f| = |\tilde{a}_f|,\ f = 1, ..., N_f.} \tag{1.20}$$

For generic $a_f \neq 0, f = 1, ...N_f$ $SU(N_c)$ breaks down to $SU(N_c - N_f)$. Now the question is what kind of quantity describes this D-flat moduli space. In other words, what degrees of freedom are the relevand ones at low energies. Since a spontaneous symmetry breaking is at work the natural guess is that of N_f^2 gauge invariant composite chiral "meson"superfields (Goldstone supermodes)

$$M^f{}_g = Q^{fi} \tilde{Q}_{gi}, \qquad f, g = 1, ..., N_f. \tag{1.21}$$

whose vacuum expectation values parametrize the vacuum manifold.

$$\langle M^f{}_g \rangle = \mathrm{diag}(a_1^2, a_2^2, ..., a_{N_f}^2). \tag{1.22}$$

The counting matches. We started with $2N_c N_f$ massless chiral superfields Q^f and $\tilde{Q}_g$, and ended up with N_f^2 modes $M^f{}_g$. The difference is the number of degrees of freedom (Goldstone modes) that have been eaten up in the Higgs mechanism, which is also the number of particles becoming massive

$$\overbrace{(N_c^2 - 1)}^{SU(N_c)} - \overbrace{(N_c - N_f)^2 - 1}^{SU(N_c - N_f)} = \overbrace{(2N_c N_f - N_f^2)}^{\text{Higgsed modes}} \tag{1.23}$$

- $\underline{N_f \geq N_c}$. In this case, the best one can do is bring the chiral superfields to the form

$$\phi^{fi} = a_i \delta^{fi} = \left.\overbrace{\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{N_f} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}}^{N_c}\right\} N_f \tag{1.24}$$

and the same for $\tilde{\phi}_{gi} = \tilde{a}_g \delta_{gi}$. In this case the D-flatness condition (1.17) is the same but now with $N_f \geq N_c$. Given the tracelesness of the gauge generators $\sum_{i=1}^{N_c} (T^a)^i{}_i = 0$, the generic solution of (1.203) is given by

$$\boxed{|a_f|^2 - |\tilde{a}_f|^2 = v^2 \ , \qquad \forall f = 1, ..., N_f} \tag{1.25}$$

Now the gauge group is *completely broken* at a generic point on moduli space. The gauge invariant condensates that can be formed are now

$$\begin{aligned} M^f{}_g &= Q^{fi} \tilde{Q}_{gi} & (1.26) \\ B^{[f_1 \dots f_{N_c}]} &= Q^{f_1 i_1} \dots Q^{f_{N_c} i_{N_c}} \epsilon_{i_1 \dots i_{N_c}} & (1.27) \\ \tilde{B}_{[f_1 \dots f_{N_c}]} &= \tilde{Q}_{f_1 i_1} \dots \tilde{Q}_{f_{N_c} i_{N_c}} \epsilon^{i_1 \dots i_{N_c}} \ . & (1.28) \end{aligned}$$

The "baryon" indices are automatically antisymmetrized in flavor space. In total they are

$$N_f^2 + 2C_{N_f}^{N_c} = N_f^2 + \frac{2N_f!}{N_c!(N_f - N_c)!} \tag{1.29}$$

quantities, which exceeds the number of initial chiral modes minus the number of broken generators $2N_c N_f - (N_c^2 - 1)$. The clue is that the previous quantities are in some cases redundant, as can be seen by finding agebraic identities that link them. Let us see this in the special cases $N_c = N_f$ and $N_c = N_f + 1$.

- $\underline{N_f = N_c}$ The matrices Q^{fi} and $\tilde{Q}_{fi}$ are square matrices, and hence $N_f^2 + 2$ gauge invariant polynomial can be formed

$$M^f{}_g = Q^f \tilde{Q}_g \ , \qquad B = \det Q^{fi} \ , \qquad \tilde{B} = \det \tilde{Q}_{fi} \ . \tag{1.30}$$

However they are linked by the following trivial identity

$$\boxed{\det M = B\tilde{B}} \tag{1.31}$$

So we have really $N_f^2 + 1$ moduli. Let us check the counting: $SU(N_c)$ is completely broken, hence $N_c^2 - 1 = N_f^2 - 1$ Goldstone modes have been eaten up. Since we started from $2N_c N_f = 2N_f^2$, the difference yields precisely $N_f^2 + 1$ moduli, which are the ones in (1.30) modulo the constraint (1.31).

- $\underline{N_f = N_c + 1}$ In this case there are $N_f^2 + 2N_f$ gauge invariant quantities It is convenient to exhibit the bayonic operators (1.27) and (1.28) in the Hodge dual form

$$\bar{B}_g = \frac{1}{N_c!}\epsilon_{gf_1\cdots f_{N_c}} Q^{f_1 i_1} Q^{f_2 i_2} \cdots Q^{f_{N_c} i_{N_c}} \epsilon_{i_1 i_2 \cdots i_{N_c}} \tag{1.32}$$

$$\bar{\tilde{B}}^g = \frac{1}{N_c!}\epsilon^{gf_1\cdots f_{N_c}} \tilde{Q}_{f_1 i_1} \tilde{Q}_{f_2 i_2} \cdots \tilde{Q}_{f_{N_c} i_{N_c}} \epsilon^{i_1 i_2 \cdots i_{N_c}} \tag{1.33}$$

$$M^f{}_g = \tilde{Q}^f Q_g \tag{1.34}$$

which exceeds by $2N_f$ the number of moduli $2N_cN_f - (N_c^2 - 1) \overset{N_c \to N_f - 1}{=} N_f^2$. Again they are not independent but satify the equalities

$$\boxed{\bar{B}_f M^f{}_g = M^f{}_g \bar{\tilde{B}}^g = 0 \quad ; \quad \det M (M^{-1})_f{}^g = \bar{B}_f \bar{\tilde{B}}^g} \tag{1.35}$$

Indeed

$$\bar{B}_f M^f{}_g = \frac{1}{N_c!}\epsilon_{ff_1\cdots f_{N_c}} Q^{f_1 i_1} Q^{f_2 i_2} \cdots Q^{f_{N_c} i_{N_c}} Q^{fj} \epsilon_{i_1 i_2 \cdots i_{N_c}} \tilde{Q}_{gj} = 0 \tag{1.36}$$

and this vanishes because the set of indices $[i_1, ..., i_{N_c}, i]$ is antisymmetric and necessarily two of them are repeated. In the second equation, the left hand side stands short for the minor of the element $M^g{}_f$ *i.e.* the determinant of M with the f'th row and g'th column deleted. This is precisely

$$\begin{aligned} l.h.s. &= \frac{1}{N_c!}\epsilon_{f_1\cdots f_{N_c} g} \epsilon^{g_1 \cdots g_{N_c} f} M^{f_1}{}_{g_1} \cdots M^{f_{N_c}}{}_{g_{N_c}} \\ &= \frac{1}{N_c!}\epsilon_{f_1\cdots f_{N_c} g} Q^{f_1 i_1} \cdots Q^{f_{N_c} i_{N_c}} \epsilon^{g_1 \cdots g_{N_c} f} Q_{g_1 i_1} \cdots Q_{g_{N_c} i_{N_c}} \\ &= \epsilon_{f_1\cdots f_{N_c} g} Q^{f_1 1} \cdots Q^{f_{N_c} N_c} \epsilon^{g_1 \cdots g_{N_c} f} Q_{g_1 1} \cdots Q_{g_{N_c} N_c} \\ &= \bar{B}_g \bar{\tilde{B}}^f = r.h.s \end{aligned} \tag{1.37}$$

1.3. SQCD, Quantum Effective Action

The vacuum is the lowest energy state of the theory. The proper tool to investigate it when quantum fluctuations are taken into account is the *quantum effective action*. There are two concepts for such object. The standard textbook in QFT introduces it as the generating functional of 1PI correlation functions. In a more modern approach the Wilsonian effective action links with the theory of critical phenomena and renormalisation group analysis. Whereas the first one is plagued with infinities and must be subject to a proper renormalisation program, the second is finite and well defined by construction, hence we will stick to it in what follows.

Defining a quantum field theory starts by giving a classical local action which describes the degrees of freedom and the classical dynamics. Central to the discussion are two concepts: the coupling constants and the cutoff scale.

Without a cutoff scale, a QFT is ill defined and plagued with infinities. The classical renormalisation program is a way to handle these divergences. Another approach is that of a Wilsonian effective action. A Wilsonian effective action has the following generic aspect

$$S(\mu) = \int d^d x \sum_i g_i(\mu)\mathcal{O}_i \tag{1.38}$$

were $\mathcal{O}_i$ label an infinite set of local operators (polynomials in basic fields and derivatives thereof)

A field theory is normally defined by specifying the bare parameters in a given action $g_i(\mu)$ at some cutoff scale μ. One then makes use of this action to seed a path integral that will produce Green's functions which will be calculable in terms of $g_i(\mu)$. Imagine $|p_\mu| \sim E \leq \mu$ is the typical (euclidean) momentum scale of the incoming particles in the process. Then we loosely refer to these amplitudes as $\Gamma(E, g_i(\mu))$. Fixing these amplitudes to experimental values is a practical means of fixing the values of the couplings $g_i(\mu)$. The actual computation of $\Gamma(E, g_i(\mu))$ involves integrating over loop momenta p_μ in the range $E \leq |p_\mu| \leq \mu$. This typically introduces logarithms $\log(\mu/E)$. Hence on one side, for $E = \mu$ the tree level result is exact, and on the other hand for $E << \mu$ the logarithms grow larger than the tree level values and spoil the perturbative analysis.

It may therefore be convenient to define the theory at a different scale $\mu' < \mu$ such that the tree level results better approximate the physics. This is the task of the renormalization group, which answers to the following question: how must the parameters change in order for the physics at scale E to remain intact?

$$\Gamma(E, g_i(\mu), \mu) = \Gamma(E, g_i'(\mu'), \mu') \tag{1.39}$$

The dependence of g_i on the defining scale μ is encoded in the Renormalization Group Equation (RGE)

$$\mu \frac{dg_i(\mu)}{d\mu} = \beta_i(g_i(\mu), \mu)) \tag{1.40}$$

As an example consider the perturbed free scalar field theory

$$S[\phi]_{free} = \int d^d x \, \frac{1}{2}\partial\phi\partial\phi \tag{1.41}$$

Clearly for this action to be invariant under $\mu \to \mu' = \lambda\mu$ and $x \to x' = \lambda^{-1}x$ where $\lambda < 0$ (hence x growing in length) ϕ must scale as $\phi \to \lambda\phi$. Hence the mass dimension $[\phi] = +1$. Consider an operator made out of ϕ and derivatives thereof such that $\mathcal{O}_i \to \lambda^{d_i}\mathcal{O}_i$. We say that $\mathcal{O}_i$ has classical (mass) dimension d_i. Next we write the perturbed action as follows

$$S[\phi; \mu, g_i] = \int d^d x \left[\frac{Z(\mu)}{2}\partial_\mu\phi\partial^\mu\phi + \sum_i g_i(\mu)\mathcal{O}_i(x) \right] \tag{1.42}$$

For the action to be dimensionless, hence invariant under the previous scaling, the coupling constants must have *classical dimension* $d - d_i$. We write $g_i(\mu) = \mu^{d-d_i}\tilde{g}(\mu)$ where $\tilde{g}(\mu)$ is dimensionless. Hence we have

$$\begin{aligned}
\mu\frac{dg_i}{d\mu} &= (d_i - d)\mu^{d_i-d}\tilde{g}_i(\mu) + \mu^{d-d_i}\mu\frac{d\tilde{g}_i(\mu)}{d\mu} \\
&= (d_i - d)g_i(\mu) + \beta_i^{quant} \\
&= \beta_i
\end{aligned} \tag{1.43}$$

Of particular importance are the limits $\mu \to \infty$ (UV) and $\mu \to 0$ (IR). If there exist a sensible limit where the couplings don't run away, then the theory must reach a fixed point $g_(\mu \to 0) = g_i^*$. Theories at a fixed point are very special because they are scale and even conformal invariant. They have neither dimension-full parameters nor massive states. They are called *conformal field theories* (CFT).

In the neighbourhood of a fixed point CFT we have $g_i = g_i^* + \delta g_i$ and we can always linearise the RG flows

$$\mu\frac{dg_i}{d\mu}\bigg|_{g_j^*+\delta g_j} = A_{ij}\ \delta g_j + \mathcal{O}(\delta g_j^2) \tag{1.44}$$

and go to a diagonal basis $\delta g_i \to \delta\tilde{g}_i$, $A_{ij} \to \Delta_i\delta_{ij}$ such that , to linear order

$$\mu\frac{d\delta\tilde{g}_i}{d\mu} = (\Delta_i - d)\delta\tilde{g}_i = \beta_i^{quant}(g^*) \tag{1.45}$$

and so, to linear order the RG flow is simply

$$\delta\tilde{g}_i(\mu) = \left(\frac{\mu}{\mu'}\right)^{\Delta_i-d}\delta\tilde{g}_i(\mu') \tag{1.46}$$

The quantity Δ_i is called the *scaling (or conformal) dimension* of the operator associated to $\delta\tilde{g}_i$. It will be different, in an interacting QFT, from the classical scaling dimension d_i, actually

$$\Delta_i = d_i + \gamma_i \tag{1.47}$$

where is called the *anomalous dimension* of the operator and its origin is purely quantum.

From (1.45) we see that, close to the fixed point, the couplings can be classified in the following way

- *Relevant.* If $\Delta_i < d$ the coupling decreases (increases) towards the UV (IR). So in the vicinity of a UV fixed point ($\mu \to \infty$) all these couplings vanish.
- *Irrelevant.* If $\Delta_i > d$ we have just the opposite of the above. Hence as we approach a IR fixed point ($\mu \to 0$) all irrelevant couplings disappear, and only relevant couplings remain.
- *Marginal.* The case $\Delta_i = d$ signals couplings which stay fixed under the RG flow. However for them, the linearised analysis is not valid and one has to go beyond leading order. If finally the coupling does not run to all orders, we speak of a *truly marginal* coupling.

In summary, when building the low energy effective action, which capture the IR physics, only relevant operators with $\beta_i < 0$ are important. Moreover, they become negligible in the UV, so the actual value they have there is actually of very little importance. The number of relevant operators is finite, and actually very small in a supersymmetric theory. From the perturbative point of view, relevant operators are the same as renormalizable ones. So those for which the coupling constants have scaling dimension $\Delta_i < 4$. In a first approximation, we can trade the full scaling dimension for the classical one $\Delta_i \sim d_i$. If we were talking about a scalar field theory this would leave us with an action of the form

$$S = \int d^4x \left(\frac{1}{2}(\partial\phi)^2 + m^2\phi^2 + \beta\phi^3 + \lambda\phi^4 \right) \tag{1.48}$$

In superspace the counting is a little bit different. This is because the anti commuting coordinates have scaling dimension $[\theta] = 1/2$ but $[d\theta] = -1/2$, given that $\int d\theta\theta = 1$.

Hence, for an action like (1.11), appart from the kinetic terms, we are speaking of superpotential terms which have at most dimension 3

$$\mathcal{W}_{rel} = m_{ij}\Phi^i\Phi^j + \lambda_{ijk}\Phi^i\Phi^j\Phi^k\,. \tag{1.49}$$

Quantum Vacua

In QFT, scalar fields may develop a non-trivial vacuum expectation value (VEV), $\langle\phi_i\rangle = 0$. The set of all VEV's characterises the vacuum state itself, and are determined by the requirement that they are minima of the *low energy effective potential*. The low energy effective potential is the potential of the WIlsonian effective action in the limit $\mu \to 0$, hence after all non-zero modes of the fields have been integrated out.

1.3.1. SQCD: Renormalization Group Analysis

From (1.11) we see that typically the supersymmetric version of QCD consists of three types of terms. Let us repeat here the tree level action, stressing the fact that it is the *UV* action defined at scale μ_0

$$\boxed{\begin{aligned} \mathcal{L}(\mu_0) &= \mathrm{Im}\left(\frac{\tau(\mu_0)}{8\pi}\mathrm{Tr}\int d^2\theta W^2\right) + Z(\mu_0)\int d^2\theta d^2\bar\theta(Q^\dagger e^{2V}Q + \tilde{Q}e^{-2V}\tilde{Q}^\dagger) \\ &\quad + \left(\int d^2\theta\, \mathcal{W}(Q,\tilde{Q};\lambda_i(\mu_0)) \;+\; h.c.\right) \end{aligned}} \tag{1.50}$$

We have displayed the parameters that will depend on the UV renormalization scale, which we have called μ_0. Actually it is conventional to set $Z(\mu_0) = 1$ by definition as an initial condition.

Perturbatively, the fate of the different terms along RG flow $\mu_0 \to \mu$ is condensed in the following collection of statements

1. The couplings $\lambda_i(\mu)$ are invariant, hence receive no corrections.

2. The function $\tau(\mu)$ receives corrections at 1-loop in perturbation theory.

3. The function $Z(\mu)$ receives corrections from all loops in perturbation theory.

1.3.1.1 Non-renormalization theorem

From these, the most famous one is the last one, which usually goes under the name of "non-renormalization theorem", stating that the superpotential is not renormalized in perturbation theory. Let us give here a proof based on symmetry arguments [2]. The prototypical example deals with the Wess-Zumino lagrangian

$$S = \int d^4\theta \, \Phi\Phi^\dagger + \int d^2 \; \theta(m\Phi^2 + \lambda\Phi^3) \, . \tag{1.51}$$

There is no $U(1)_R$ charge assignement that makes this lagrangian invariant. However one can play the trick of envisaging the constants m, λ as background (spurion) fields that have acquired a v.e.v. The following charges do the job

	$U(1)_R$	$U(1)_B$
Φ	1	1
m	0	-2
λ	-1	-3

(1.52)

Under RG flow, the Kähler form and the superpotential will run over to some unknown functions

$$\begin{aligned} \mathcal{K} &= \mathcal{K}(\Phi, \Phi^\dagger, m, m^*, \lambda, \lambda^*) \\ \mathcal{W} &= \mathcal{W}(\Phi, m, \lambda) \end{aligned} \tag{1.53}$$

In a Wilsonian setup, integrations over finite momentum intervals cannot spoil holomorphicity of $\mathcal{W}$. This, together with the requirement that the effective lagrangian has the same global symmetries restricts the form of $\mathcal{W}$ as

$$\mathcal{W} = m\Phi^2 f\left(\frac{\lambda\Phi}{m}\right) = \sum_n a_n \lambda^n m^{1-n} \Phi^{n+2} \tag{1.54}$$

The existence of a weak coupling limit $\lambda \to 0$ restricts $n \geq 0$, and continuity of the massless limit $m \to 0$ implies $n \leq 1$, thus

$$\mathcal{W}(\mu) = m\Phi^2 + \lambda\Phi^3 = \mathcal{W}(\mu_0) \tag{1.55}$$

1.3.1.2 Perturbative Non-Renormalization theorem

Perturbation theory can be implemented in full superspace, by means of the supergraph computation. Each such supergraph computes a contribution to the $1PI$ Effective Action. An important point to stress, is the fact that all such contributions come out with an integral over full superspace $\int d^2\theta d^2\bar{\theta} d^4 p$..... In other words, superspace perturbation theory only produces D terms, and never F terms. Hence it will be only capable of giving contributions that correct the Kähler potential, but never the superpotential. So simple is in essence the content of the famous nonrenormalization theorem. Of course this only means that the parameters in the superpotential $\tau, m^2, \lambda, ...$ do not have renormalizations Z's, other that the ones than come from the wave function renormalization Z_Q and $Z_{\tilde{Q}}$.

1.3.1.3 *Holomorphic Anomalies*

In supersymmetric gauge theories and, in general, in theories with massless particles that can propagate inside the loops, some propagators may lead to infrarred singularities in the limit $k^\mu \to 0$. This offers a mechanism to evade the perturbative non-renormalization theorem. Consider the following D-looking-term

$$I = \int d^4x d^2\theta d^2\bar{\theta} \frac{D^2}{\Box} F(\Phi) \tag{1.56}$$

By using the identity $[D^2, \bar{D}^2] \sim \Box$ we may perform the following manipulations, (neglect total derivatives)

$$\begin{aligned} I &= \int d^4x \left(\frac{1}{16} D^\alpha \bar{D}^2 D_\alpha - \frac{1}{4}\Box\right) \frac{D^2}{\Box} F(\Phi)\Big|_{\theta=\bar{\theta}=0} \\ &= \frac{1}{16} \int d^4x D^\alpha \frac{\bar{D}^2 D^2}{\Box} D_\alpha F(\Phi)\Big|_{\theta=\bar{\theta}=0} \\ &= \frac{1}{16} \int d^4x D^\alpha \frac{[\bar{D}^2, D^2]}{\Box} D_\alpha F(\Phi)\Big|_{\theta=\bar{\theta}=0} \\ &= \frac{1}{16} \int d^4x D^2 F(\Phi)\Big|_{\theta=\bar{\theta}=0} \\ &= -\frac{1}{4} \int d^4x d^2\theta F(\Phi) \end{aligned} \tag{1.57}$$

The end result goes into a correction of the superpotential. However this situation is to be expected only in 1PI effective actions. In Wilsonian effective actions the integrals in momentum are cutoff, above by the UV definition scale μ_0 and below by the renormalization scale μ. In conclusion, they never will generate a singular contribution like (1.56). Thus, we state that a *Wilsonian superpotential is never generated in pertubation theory.* At the non-perturbative level, *i.e.* functions that admit an expansion in powers of e^{1/g^2}, the possible answers are constrained by the global symmetries. Flavour symmetries $SU(N_f) \times SU(N_f)$ impose to use as variables singlets like $\det M$, B and $\tilde{B}$. $U(1)$ symmetries are more delicate, since care of chiral anomalies must be taken.

1.3.1.4 *Anomalous dimensions from wave function renormalization*

It is important to stress that the non-renormalization theorem for the couplings λ_i in the superpotential has been derived under the assumption that the kinetic term is not canonically normalized. Contrarily to the superpotential, the Kähler potential, not being protected by holomorphy can, and indeed will, receive contributions from higher loops that will correct it multiplicatively

$$\mathcal{K}(\mu) = Z(m, m^*, \lambda, \lambda^*, \mu)\mathcal{K}(\mu_0). \tag{1.58}$$

In order to compare theories along the RG trajectory it is always convenient to stick to normalized kinetic terms for all the fields. In particular, for the chiral superfields this involves a redefinition

$$Q = \frac{1}{\sqrt{Z(\mu,\mu_0)}} Q_c \quad ; \quad \tilde{Q} = \frac{1}{\sqrt{Z(\mu,\mu_0)}} \tilde{Q}_c \,. \tag{1.59}$$

In terms of these $(Q_c, \tilde{Q}_c)$ the kinetic terms will always stay canonically normalized

$$\begin{aligned} \mathcal{L}(\mu) &= \mathrm{Im}\left(\frac{\tau(\mu)}{8\pi}\mathrm{Tr}\int d^2\theta W^2\right) + \int d^2\theta d^2\bar{\theta}\left(Q_c^\dagger e^{2V} Q_c + \tilde{Q}_c e^{-2V}\tilde{Q}_c^\dagger\right) \\ &+ \;\; (\int d^2\theta\, \mathcal{W}(Q,\tilde{Q};\lambda_i^c(\mu)) \;+\; h.c.) \end{aligned} \tag{1.60}$$

(1.61)

However, now the new couplings are no more RG invariant. For example for the mass term $mQ\tilde{Q} = m^c Q_c \tilde{Q}_c$ we would have

$$m^c(\mu) = m\, Z(\mu,\mu_0)^{-1} \tag{1.62}$$

and from here the anomalous dimension totally arises from the wave function renormalization

$$\beta_{m^c} \equiv \frac{dm^c(\mu)}{d\log\mu} = -m^c(\mu)\frac{d\log Z(\mu,\mu_0)}{d\log\mu} \equiv -m^c(\mu)\gamma(\mu) \tag{1.63}$$

1.3.1.5 *Perturbative Quantum Moduli Space*

In this way we arrrive at the following important result: if we start from a bare lagrangian which has a vanishing superpotential $\mathcal{W} = 0$, a Wilsonian superpotential will never be generated along a RG trajectory in perturbation theory.

Stated in a different way, the classical moduli space and the quantum moduli space, given by solving the D flatness condition (1.203), are the same.

Reverting the logic of the argument, if a superpotential is seen to arise along an RG-trajectory, it must necessarily come from non-perturbative effects, such as instanton configurations, which introduce corrections in the effective action weighted by e^{1/g^2}. The possible superpotentials thus generated dynamically are only constrained by the global unbroken symmetries. Flavor symmetries $SU(N_f)_L \times SU(N_f)_R$ dictate that the allowed superpotentials must be functions of flavor singlet polynomials $\det(M)$, B or $\tilde{B}$. $U(1)$ symmetries are more delicate as they involve chiral fermions and care must be taken of anomalies. The main aim of these lectures is the study of non-perturbatively generated superpotentials.

1.3.2. Holomorphic β function

A Wilsonian UV action at scale μ incorporates the information about all the frequencies at scales above μ. Hence it is only capable of describing physical quantities at lower energy scales $E \leq \mu$. The computation will tipically involve integrals ranging from μ to 0, but the slice between E and 0 can be included in a renormalization scheme. In this scheme, the tree level form of the UV Wilsonian action at scale μ describes the physical processes at energy $E \simeq \mu$, up to corrections of $\mathcal{O}(\log \mu/E)$. The change of the parameters when $\mu \to \mu'$ by construction keeps the physics at $E \leq \mu' \leq \mu$ invariant.

If we take the action (1.50) as the defining UV action at scale μ_0, at a lower scale $\mu < \mu_0$, the form of the action is the same, with renormalized $\tau_0 = \tau(\mu_0) \to \tau(\mu)$.

$$\begin{aligned}\mathcal{L}(\mu) &= \mathrm{Im}\left(\frac{\tau(\mu)}{8\pi}\mathrm{Tr}\int d^2\theta W^2\right) + Z(\mu)\int d^2\theta d^2\bar{\theta}(Q^\dagger e^{2V}Q + \tilde{Q}e^{-2V}\tilde{Q}^\dagger) \\ &+ \left(\int d^2\theta\, \mathcal{W}(Q,\tilde{Q};\lambda_i(\mu_0)) \ + \ h.c.\right) \end{aligned} \tag{1.64}$$

where we have taken account of the non-renormalization theorem for the superpotential automatically. Under and RG flow, $\mu_0 \to \mu < \mu_0$ the gauge coupling constant gets renormalized as

$$\tau(\mu) = \tau(\mu_0) + f(\tau_0;\mu,\mu_0)\,. \tag{1.65}$$

In order to get hold on the possible form for $f(\tau_0;\mu,\mu_0)$ we take into consideration the following constraints

- *holomorphicity:* by which we mean that $f(\tau_0;\mu,\mu_0)$ is a holomorphic function of τ_0.
- *periodicity:* under shifts $\theta \to \theta + 2\pi i \Rightarrow \tau_0 \to \tau_0 + 1$. Then at most $f(\tau_0+1;\mu,\mu_0) = f(\tau_0;\mu,\mu_0) + n(\mu,\mu_0)$ where $n(\mu,\mu_0) \in \mathbb{Z}$. Given the boundary condition $n(\mu_0,\mu_0) = 0$ by continuity we arrive at $n(\mu,\mu_0) = 0$. This means $f(\tau_0;\mu,\mu_0)$ is a periodic function under shifts $\theta \to \theta + 2\pi i$, or $f(\tau_0+1;\mu,\mu_0) = f(\tau_0;\mu,\mu_0)$ and we may expand in Fourier series

$$\tau(\tau_0;\mu,\mu_0) = \tau_0 + \sum_{n=0}^{\infty} f_n\left(\frac{\mu}{\mu_0}\right) e^{2\pi n i \tau_0} \tag{1.66}$$

- *transitivity:* $f_0(\mu',\mu_0) = f_0(\mu',\mu) + f_0(\mu,\mu_0)$. This forbids to a logarithm $f_0(\mu,\mu_0) = \log(\mu/\mu_0)$.

In summary, fulfilling all the constraints yields an almost unique answer, and we get the important result that the perturbative contribution is saturated at one loop

$$\tau_{pert}(\frac{\mu}{\mu_0},\tau_0) = \tau_0 \ + \ \frac{ib}{2\pi}\ln\frac{\mu}{\mu_0}\ . \tag{1.67}$$

and with $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$, we obtain the perturbative running of the coupling constant

$$\boxed{\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \frac{b}{8\pi^2}\log\frac{\mu}{\mu_0}} \tag{1.68}$$

where b is the coefficient of the 1-loop beta function in the usual RG equation

$$\mu\frac{dg(\mu)}{d\mu} = -\frac{b}{16\pi^2}g^3 \tag{1.69}$$

which must be computed explicitely. One may infere it from the standard QCD result and the field content in (1.50).

$$b = \frac{11}{6}T(A) - \frac{1}{3}\sum_{fer} T(R_{fer}) - \frac{1}{6}\sum_{sc} T(R_{sc}) \tag{1.70}$$

where "fer" runs over Weyl fermions and "fer" over complex scalars. In $N = 1$ SYM, "fer" will count the gauginos λ in the adjoint ($R = A$) and the $2N_f$ Weyl quarks $\psi^f, \tilde{\psi}_f$,, whereas "sc"will run over the $2N_f$ squarks $\phi^f, \tilde{\phi}_f$. Therefore we find

$$\begin{aligned} b &= \left(\frac{11}{6} - \frac{1}{3}\right) T(A) - \sum_{f=1}^{2N_f} \left(\frac{1}{3} + \frac{1}{6}\right) T(R_f) \\ &= \frac{3}{2}T(A) - \sum_{f=1}^{N_f} T(R_f) \end{aligned} \tag{1.71}$$

where $T(R) = C(R)/C(F)$ is the index of the representation R, and $C(R)\delta^{ab} = \mathrm{tr}(R(t^a)R(t^b))$. Tipically $C(F) = 1/2$ and $C(A) = N$ for $SU(N)$, hence

$$T(R) = \begin{cases} 1 & R = F = N_c \text{ or } \bar{N}_c \text{ of } SU(N_c) \\ 2N_c & R = A = \text{adjoint of } SU(N_c) \end{cases} \tag{1.72}$$

and so we obtain for b

$$\boxed{b = 3C(A) - \sum_{f}^{N_f} C(R_f) = 3N_c - N_f \ .} \tag{1.73}$$

if all the matter falls in the fundamental representation. From (1.68) we derive two known results.

- Differentiating with rexpect to $\log\mu$ we obtain the *holomorphic perturbative beta function* for $SU(N_c)$ with N_f flavours

$$\boxed{\beta(g) = -\frac{g^3}{16\pi^2}(3N_c - N_f)} \tag{1.74}$$

- Defining $\mu = \Lambda$ as the scale where $g(\Lambda) = \infty$ gives

$$\boxed{\Lambda^b = \mu_0^b \exp\left(-\frac{8\pi^2}{g^2(\mu_0)}\right) .} \tag{1.75}$$

  Making use of (1.69) with $\mu = \mu_0$ one can easily check that $d\Lambda/d\mu_0 = 0$. Hence, Λ *does not depend on* μ_0 itself, but rather *on the value of the bare coupling* $g(\mu_0)$ at the UV scale μ_0. This trading of a dimensionless quantity, $g(\mu_0)$, by a dimensionfull number, Λ, goes under the name of *dimensional transmutation*. It will turn out to be consistent to extend the definition of Λ to complex values,

$$\boxed{\left(\frac{\Lambda}{\mu_0}\right)^b = \exp\left(-\frac{8\pi^2}{g^2(\mu_0)} + i\theta\right) = e^{2\pi i\tau(\mu_0)}} \tag{1.76}$$

<u>*1.3.2.1*</u> *Absence of non-perturbative corrections for pure $N = 1$ SYM*

We have seen in (1.67) that the perturbative running is exhausted by the one-loop contribution. In the case of pure $N = 1$ SYM, A slight refinement of this argument permits to show that potential non-perturbative corrections shown in (1.66) are also absent. In $N = 1$ SYM there are massless Weyl fermions, and as we will soon see, anomalous global $U(1)$ symetries. In such a context, the θ parameter is as unphysical as the choice of the origin of phases for the Weyl fermions. Therefore there should be a consistent reduction of the RG flow to $\theta = 0$. However it is evident from (1.66) that setting $\text{Re}(\tau_0) = 0$ we do not get $\text{Re}\tau(\mu) = 0$ except if $f_n = 0$ for $n \geq 1$. In contrast, the $n = 0$ piece (1.67) leaves θ intact.

1.3.3. Matching Conditions

The previous analysis allows us to explore the behaviour of the effective action when the scale is moved relative to the value of the relevant scales in the theory. The later are set by dimensionfull couplings, like masses, and by vevs.

Consider the following two possible scenarios

- Suppose we start from an $SU(N_c)$ theory with N_f flavors. From (1.203) we learn that we can give to one of the chiral fields, a *vev*, say $|a_{N_f}| = |\tilde{a}_{N_f}| = v$. With large momentum scales $q >> v$ the theory looks like $SU(N_c)$ with N_f flavors, but for $q << v$ the theory looks like $SU(N_c - 1)$ with $N_f - 1$ flavors. Each regime has its own dynamical scale Λ. This setup can be generalized to an arbitrary number of vevs. $|a_g| = |\tilde{a}_g| = v$ for $g = 1, ..., \Delta$).

- Alternatively, we may imagine that a mass term for a given quark is introduced

  $$\mathcal{L}_m = m\tilde{Q}Q$$

  For energies $p >> m$ the theory looks like $(SU(N_c), N_f)$. However much below $p << m$ the quanta of this field are not excited and we may replace it by its classical *vev*, which actually is vanishing, as imposed by the F term conditions:

  $$F_Q = m\tilde{Q} = 0 \quad ; \quad F_{\tilde{Q}} = mQ = 0$$

  Therefore the field completely disappears, and the theory looks effectively like $(SU(N_c), N_f - 1)$.

To cope with both possibilities consider the general problem of matching of high and low energy theories

$$(SU(N_c), N_f) \overset{s}{\hookrightarrow} (SU(\tilde{N}_c), \tilde{N}_f) \tag{1.77}$$

with $\tilde{N}_c = N_c - \Delta$ and $\tilde{N}_f = N_f - \Xi$, where s stands for a mass scale, either v or m, where both, high and low energy-theories must connect. Each theory has its own RG equation dynamical scale

$$\begin{aligned}\Lambda(\mu) &= \mu e^{\frac{2\pi i\tau(\mu)}{b}} \\ \tilde{\Lambda}(\tilde{\mu}) &= \tilde{\mu} e^{\frac{2\pi i\tilde{\tau}(\tilde{\mu})}{\tilde{b}}}\end{aligned} \tag{1.78}$$

In principle, the definition scales μ and $\tilde{\mu}$ are independent. However in this case they are linked by the fact that the effective theory and the microscopic theory match together at scale s

$$\boxed{\tau(s) = \tilde{\tau}(s) \quad \Rightarrow \quad \Lambda^b = \tilde{\Lambda}^{\tilde{b}} s^{b-\tilde{b}}} \tag{1.79}$$

1.3.4. Konishi Anomaly

The Konishi anomaly is the supersymmetric version of the familiar axial anomaly. Consider to the following super-chiral transformations $Q \to Q'$ with

$$\begin{aligned}Q'^f = e^{i\alpha} Q^f \qquad &; \qquad \tilde{Q}'_f = e^{i\beta}\tilde{Q}_f \\ Q'^{\dagger f} = e^{-i\alpha}\bar{Q}^f \qquad &; \qquad \tilde{Q}'^{\dagger}_f = e^{-i\beta}\tilde{Q}^{\dagger}_f\end{aligned} \tag{1.80}$$

with α, β real parameters. This is by itself a (global) symmetry of the ungauged action

$$\begin{aligned}S(Q, Q^\dagger, \tilde{Q}, \tilde{Q}^\dagger; V) &= \int d^4x d^4\theta \; (Q^\dagger e^{2V} Q + \tilde{Q} e^{-2V} \tilde{Q}^\dagger) \\ &+ \int d^4x d^2\theta \mathcal{W}(Q, \tilde{Q}) + \int d^4x d^2\theta \bar{\mathcal{W}}(Q^\dagger, \tilde{Q}^\dagger)\end{aligned} \tag{1.81}$$

for vanishing superpotential $\mathcal{W} = 0$. Considering the action of this symmery on the fermions ψ and $\tilde{\psi}$ inside Q^f and $\tilde{Q}^f$ for $\alpha = \beta$ this is an extension of the familiar $U(1)_A$ axial symmetry, whereas for $\alpha = -\beta$ this is the analog of the vector (or baryon) $U(1)_V$ symmetry.

Now in order to capture the associated Ward identity we promote this global transformations to local ones. Moreover we will relax the reality conditions on α and β and promote them to independent chiral superfields $\alpha(x, \theta, \bar{\theta})$ and $\beta(x, \theta, \bar{\theta})$. Then the classical action will no more be invariant (unless we not transform the gauge field to compensate for it, in which case we are actually gauging the symmetry). An additional ingredient is that we shall treat all chiral fields $Q, Q^\dagger, \tilde{Q}$ and $\tilde{Q}^\dagger$ as independent from one another (notice that, for example, the scalar ϕ in Q is now complex, hence we have doubled the degrees of freedom). Associated with each independent transformation of each field, we shall find an anomalous Ward identity.

For example, increasing just

$$Q \to e^{i\alpha(x,\theta,\bar{\theta})} Q \tag{1.82}$$

leads to [2]

$$\begin{aligned}\delta_\alpha S &= i\alpha(x,\theta,\bar\theta)\frac{\delta S(e^{i\alpha}Q)}{\delta\, i\alpha(x,\theta,\bar\theta)}\\ &= i\alpha(x,\theta,\bar\theta)\left(-\frac{1}{4}\bar D^2 Q_f^\dagger e^{2V}Q^f + Q^f\frac{\partial\mathcal{W}(Q,\tilde Q)}{\partial Q^f}\right)\end{aligned}\tag{1.84}$$

Making $\alpha\in R$ global does not make the action invariant, so we may not derive a classical conservation equation [3]. However the above transformation (1.82) is by itself a quantum symmetry of the path integral $Z(Q,Q^\dagger,\tilde Q,\tilde Q^\dagger)$ since the quantum field Q is integrated over and (1.82) is just a change of variables. The measure of integration changes by a jacobian,

$$(DQ^f)\to(De^{i\alpha}Q^f)=(DQ^f)J(\alpha)\tag{1.85}$$

whose covariant-superspace evaluation was performed by Konishi and Shizuya [4], with the result

$$\boxed{J(i\alpha)=\exp\left(-\int d^4x d^2\theta\frac{T(R_f)}{32\pi^2}\,(i\alpha)\ \mathrm{Tr}W^2\right)}\tag{1.86}$$

Demanding that Z be independent of α, $Z(e^{i\alpha}Q)=Z(Q)$ leads now to the Q-Ward identity

$$0=\left\langle -\frac{1}{4}\bar D^2\, Q_f^\dagger e^{2V}Q^f + Q^f\frac{\partial\mathcal{W}(Q,\tilde Q)}{\partial Q^f} - \sum_f\frac{T(R_f)}{32\pi^2}\ \mathrm{Tr}W^2\right\rangle\tag{1.87}$$

the $Q^\dagger\to e^{-i\alpha^\dagger(x,\theta,\bar\theta)}Q_f^\dagger$ gives the Jacobian

$$\boxed{\bar J(-i\alpha^\dagger)=\exp\left(-\int d^4x d^2\theta\frac{T(R_f)}{32\pi^2}\,(-i\alpha^\dagger)\ \mathrm{Tr}\bar W^2\right)}\tag{1.88}$$

and from here the $Q^\dagger$ Ward identity

$$0=\left\langle -\frac{1}{4}D^2\, Q_f^\dagger e^{2V}Q^f + Q_f^\dagger\frac{\partial\bar{\mathcal{W}}(Q^\dagger,\tilde Q^\dagger)}{\partial Q_f^\dagger} - \sum_f\frac{T(R_f)}{32\pi^2}\ \mathrm{Tr}\bar W^2\right\rangle\tag{1.89}$$

And similarly with $\tilde Q_f$ and $\tilde Q^{\dagger f}$

$$0 = \left\langle -\frac{1}{4}\bar D^2\, \tilde Q_f e^{-2V}\tilde Q^{\dagger f} + \tilde Q_f\frac{\partial\mathcal{W}(Q,\tilde Q)}{\partial\tilde Q_f} - \sum_f\frac{T(R_f)}{32\pi^2}\ \mathrm{Tr}W^2\right\rangle\tag{1.90}$$

$$0 = \left\langle -\frac{1}{4}D^2\, \tilde Q_f e^{-2V}\tilde Q^f + \tilde Q_f^\dagger\frac{\partial\bar{\mathcal{W}}(Q^\dagger,\tilde Q^\dagger)}{\partial\tilde Q_f^\dagger} - \sum_f\frac{T(R_f)}{32\pi^2}\ \mathrm{Tr}\bar W^2\right\rangle\tag{1.91}$$

[2] The $\bar D^2$ appears from the chiral functional differentiation rule

$$\frac{\delta}{\delta\alpha(x,\theta,\bar\theta)}\alpha(y,\theta',\bar\theta') = -\frac{1}{4}\bar D^2\delta(x-y)\delta(\theta-\theta')\delta(\bar\theta-\bar\theta')\tag{1.83}$$

[3] this would need a compensating transformation, i.e. $\alpha=\bar\alpha=\pm\beta=\pm\bar\beta$ for a full global vector/axial transformation

1.3.4.1 Konishi Anomaly and Gaugino Condensate

Set $\mathcal{W}(Q,\tilde{Q}) = m^f{}_g Q^g \tilde{Q}_f$ in (1.87) and $T(R_f) = T(F) = 1$ for the fundamental representation. Then it can be written in components, the lowest one being:

$$\langle Q_f^\dagger e^{2V} Q^f \Big|_{\bar{\theta}^2} \rangle = m^f{}_g \langle \tilde{\phi}_f \phi^g \rangle - \frac{1}{32\pi^2} \langle \lambda\lambda \rangle \tag{1.92}$$

It turns out that the left hand side con be expressed as a total SUSY tranformation

$$\langle Q_f^\dagger e^{2V} Q^f \Big|_{\bar{\theta}^2} \rangle = \langle \frac{1}{2\sqrt{2}} \{ \bar{Q}_{\dot{\alpha}}, \bar{\psi}^{\dot{\alpha}\, f} \phi_f(x) \} \rangle \tag{1.93}$$

In the absence of spontaneous symmery breaking $Q|0\rangle = \bar{Q}|0\rangle = 0$ leading to the following relationship among composite operator *vev*'s

$$\boxed{\langle \lambda\lambda \rangle = 32\pi^2 m^f{}_g \langle \tilde{\phi}_f \phi^g \rangle} \tag{1.94}$$

This equations states in short that, as soon as some of the scalar fields acquire a *vev* in a massive theory they will trigger a gluino condensation. Moreover (1.94) will be exact to all orders in perturbation theory.

1.3.4.2 $U(1)_A$ axial anomaly

Again setting $\mathcal{W}(Q,\bar{Q}) = \tilde{Q}_g m^f{}_g Q^g$, multiplying (1.87) by D^2 and (1.91) by $\bar{D}^2$ and subtracting one gets[4]

$$0 = \langle \partial^\mu \Gamma^5_\mu - iM - a \rangle \tag{1.96}$$

with

$$\begin{aligned} \Gamma^5_\mu &= (D\sigma_\mu \bar{D} - \bar{D}\bar{\sigma}_\mu D) \left(Q^\dagger e^{2V} Q \right) /4 \\ M &= -(D^2(\tilde{Q}_f m^f{}_g Q^g) - \bar{D}^2(Q_f^\dagger m^f{}_g \tilde{Q}^{\dagger f} \\ a &= -\frac{iN_f}{32\pi^2}(D^2 \mathrm{Tr} W^2 - \bar{D}^2 \mathrm{Tr} \bar{W}^2) \end{aligned} \tag{1.97}$$

Equation (1.96) contains the full set of anomalies forming a supermultiplet of anomalies. For example the lowest component reduces to the standard axial $U(1)_A$ anomaly

$$\partial_\mu (\bar{\psi}_f \bar{\sigma}^\mu \psi^f) = \tilde{\psi}_f m^f{}_g \psi^g + \frac{N_f}{32\pi^2} \mathrm{Tr} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \tag{1.98}$$

Another way to see this is to perform simultaneous rotation of Q and $Q^\dagger$ with real constant superfields $\alpha = \alpha^\dagger = \alpha^*$ The measure now changes by

$$(De^{i\alpha} Q^f)(De^{-i\alpha} Q_f^\dagger) = (DQ^f)(DQ_f^\dagger) J(i\alpha) \bar{J}(-i\alpha) \tag{1.99}$$

[4] one must make use of

$$[D^2, \bar{D}^2] = -i\partial_\mu (D\sigma^\mu \bar{D} - \bar{D}^\dagger \bar{\sigma}^\mu D) \tag{1.95}$$

where the total Jacobian now is

$$
\begin{aligned}
J(i\alpha)\bar{J}(-i\alpha) &= \exp\left(-i\alpha\frac{T(R_f)}{32\pi^2}\int d^4x(d^2\theta\ \mathrm{Tr}W^2 - d^2\bar{\theta}\ \mathrm{Tr}\bar{W}^2)\right) \\
&= \exp\left(\frac{\mathrm{Im}}{8\pi}\frac{\alpha T(R_f)}{2\pi}\int d^4x d^2\theta\ \mathrm{Tr}W^2\right) \qquad (1.100)
\end{aligned}
$$

Bearing in mind the form of the gauge kinetic term in the action (1.50) with $\tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta}{2\pi}$, this can be interpreted as a shift in the theta angle

$$
\theta \to \theta - \alpha T(R_f) \qquad (1.101)
$$

Notice that this result can be blamed entirely to standard axial anomaly of the Weyl fermion $\psi_{f\alpha}$ in Q_f. For a simultaneous rotation of Q and $\tilde{Q}$ the shift is by 2α.

Non-holomorphic schemes for $\beta(g)$.

1. *Canonically Normalized Chiral Superfields*

We pause here to stress that the running given in (1.68) has been obtained for a parametrization of the Wilsonian RG as given in (1.64). Let us write the effective lagrangian at scale μ (and for $\theta = 0$

$$
\begin{aligned}
\mathcal{L}(\mu) &= \frac{1}{4}\frac{1}{g^2(\mu)}\mathrm{Tr}\int d^2\theta W^2 + h.c. \\
&\quad + Z(\mu,\mu_0)\int d^2\theta d^2\bar{\theta}(Q^\dagger e^{2V}Q + \tilde{Q}e^{-2V}\tilde{Q}^\dagger) \qquad (1.102)
\end{aligned}
$$

where $\frac{1}{g^2}(\mu) = \mathrm{Re}\,\tau(\mu)$, sometimes called, holomorphic coupling constant, runs exactly at one loop

$$
\begin{aligned}
\frac{1}{g^2(\mu)} &= \frac{1}{g^2(\mu_0)} + \frac{b}{8\pi^2}\log\frac{\mu}{\mu_0} \\
\beta(\frac{1}{g^2}) &= \frac{b}{8\pi^2} \quad \Rightarrow \quad \beta(g) = -\frac{g^3}{16\pi^2}b \qquad (1.103)
\end{aligned}
$$

It is important to stress that the one-loop saturation of the RG flow for $1/g^2$ is valid only if the Kähler term undergoes the wave function renormalization $Z(\mu_0) \to Z(\mu,\mu_0)$. The Kähler potential is not a holomorphic quantity, and therefore Z is not protected. Generically it will receive contributions at all orders in perturbation theory. We may however decide to keep, at each stage in the RG trajectory, normalized kinetic terms for the chiral superfields. This amounts to a field redefinition

$$
Q_c = \sqrt{Z(\mu,\mu_0)}\,Q \quad , \quad \tilde{Q}_c = \sqrt{Z(\mu,\mu_0)}\,\tilde{Q}. \qquad (1.104)
$$

This is of the form (1.80) with $i\alpha = -i\alpha^\dagger = -\frac{1}{2}\log Z(\mu,\mu_0)$. Hence $\alpha(x,\theta,\bar\theta)$ is a purely imaginary chiral superfield. The quantum measure changes accordingly by

$$\begin{aligned} D(Q)D(Q^\dagger) &= D(e^{i\alpha}Q_c)D(e^{-i\alpha^\dagger}Q_c^\dagger) = D(Q_c)D(Q_c^\dagger)J(i\alpha)\bar J(-i\alpha^\dagger) \\ &= D(Q_c)D(Q_c^\dagger)\exp\left[-\frac{T(R_f)}{32\pi^2}\int d^4x\left(i\alpha\int d^2\theta \mathrm{Tr}W^2 - i\alpha^\dagger\int d^2\bar\theta\mathrm{Tr}\bar W^2\right)\right] \\ &= D(Q_c)D(Q^\dagger_{c\,f})\exp\left[\frac{1}{4}\frac{T(R_f)}{16\pi^2}\log Z(\mu,\mu_0)\left(\mathrm{Tr}\int d^4x d^2\theta\ W^2 + h.c.\right)\right] \end{aligned}$$

Doing the same for $\tilde Q$ and $\tilde Q^\dagger$ and adding up contributions, comparing with (1.102) we find that, at scale μ, we can have a canonically normalized lagrangian

$$\mathcal{L}_c(\mu) = \frac{1}{4}\left(\frac{1}{g_c^2(\mu)}\right)\mathrm{Tr}\int d^2\theta W^2 + h.c. + \frac{1}{4}\int d^2\theta d^2\bar\theta(Q_c^\dagger e^{2V}Q_c + \tilde Q_c e^{-2V}\tilde Q_c^\dagger) \quad (1.105)$$

with a non-holomorphic guage coupling

$$\boxed{\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \frac{b}{8\pi^2}\log\frac{\mu}{\mu_0} + 2\sum_f\frac{T(R_f)}{16\pi^2}\log Z(\mu,\mu_0)} \quad (1.106)$$

From here and, using (1.103) and (1.71) we arrive at the following non-holomorphic (all loop) beta function

$$\beta(\frac{1}{g^2}) = b - \sum_f \gamma_f = \frac{1}{8\pi^2}\left(\frac{3}{2}T(A) - \frac{1}{2}\sum_f T(R_f)(1-\gamma_f)\right) \quad (1.107)$$

with

$$\boxed{\gamma_f(\mu) = \mu\frac{\partial \log Z_f(\mu,\mu_0)}{\partial\mu}} \quad (1.108)$$

For the case of $SU(N_c)$ with $R_f = F \quad \forall f = 1,...,N_f$ we arrive finally at the following result

$$\boxed{\beta_c(g_c) = -\frac{g_c^3}{16\pi^2}\left(3N_c - N_f\left(1-2\gamma\right)\right)} \quad (1.109)$$

2. NVSZ scheme: exact beta function

This scheme is most similar to the one used for $1PI$ effective actions. It not only involves canonically normalized kinetic terms for the chiral fields, but also for the gauge fields. In components, the lagrangian (1.105) starts as follows

$$\mathcal{L}_c(\mu) = -\frac{1}{4g(\mu)}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{i}{g^2(\mu)}\lambda^a_\alpha(\sigma^\mu)_{\alpha\dot\alpha}\mathcal{D}^{ab}_\mu\bar\lambda^{\dot\alpha b} + ... \quad (1.110)$$

Hence the kinetic terms of gauge bosons and gauginos are not canonically normalized. "Canonical normalization"means that the coupling constant is not present in front of the kinetic terms, and shows up in the covariant derivatives. This leads to the usual Feynman rules, with coupling constants associated to vertices, instead of propagators. In our case, one would naively think that canonical normalization can be easily achieved through the following rescaling redefinition $V = gV_c$. Anticipating that this guess is wrong, we will replace it by another one

$$V = g_c V_c \tag{1.111}$$

and try to find the relation between g and g_c. Inocent as it looks, this change of variables is again anomalous. In fact, looking at (1.1) it is easy to recognize the action of (1.111) on the component fields

$$A^a_\mu = e^{i\beta} A^a_{c\,\mu} \;,\; \lambda^a_\alpha = e^{i\beta}\, \lambda^a_{c\,\alpha} \;,\; \bar{\lambda}^{a\,\dot{\alpha}} = e^{-i\bar{\beta}} \lambda^{a\,\dot{\alpha}}_c \;,\; D^a = e^{i\beta} D^a_c \tag{1.112}$$

with $i\beta = \log g_c(\mu) = -i\beta^*$. This again looks like a (complexified) chiral tranformation in which the Weyl fermions transform with charge 1(though the parameter has been continued to the imaginary axis). Unfortunately the trick that worked before cannot be used here, as ther is no way to express the (complex) transformation of the gluinos as a complex chiral rotation of hte vector multiplet (which is real by definition). We simply quote the result for how the measure transforms

$$\begin{aligned} (DV) &= (Dg_c V_c) = (De^{\log g_c} V_c) \\ &= (DV_c) \exp\left(\frac{T(A)}{32\pi^2} \log g_c^2(\mu) \int d^4x (d^2\theta \mathrm{Tr} W^2 + h.c.)\right) \end{aligned} \tag{1.113}$$

Notice that this has the same form as an adjoint-valued chiral multiplet anomaly, but with the opposite sign, and $i\alpha = \log g_c$. With all these ingredients let us proceed

$$\begin{aligned} Z &= \int (DV)\ \exp\left(-\frac{1}{4}\int d^4y d^2\theta \frac{1}{g^2(\mu)} \mathrm{Tr} W^2(V) + h.c.\right) \\ &= \int (Dg_c V_c)\ \exp\left(-\frac{1}{4}\int d^4y d^2\theta\ \frac{1}{g^2(\mu)}\ \mathrm{Tr} W^2(g_c V_c) + h.c.\right) \\ &= \int (DV_c)\ \exp\left(-\frac{1}{4}\int d^4y d^2\theta \left(\frac{1}{g^2(\mu)} - \frac{T(A)}{16\pi^2}\log g_c^2(\mu)\right) \mathrm{Tr} W^2(g_c V_c) + h.c.\right) \end{aligned}$$

Canonical normalization will be achieved if the whole prefactor turns out to be exactlyt $g_c^2(\mu)$, or

$$\boxed{\frac{1}{g_c^2(\mu)} = \frac{1}{g^2(\mu)} + \frac{T(A)}{16\pi^2} \log \frac{1}{g_c^2(\mu)}} \tag{1.114}$$

This important equation, sometimes known as the Shifman-Veinstein equation, performs the change of variables $g_c = g_c(g)$. With it one may compute $Z(\mu, \mu_0; g_c)$, and using (1.106)

$$\frac{1}{g_c^2(\mu)} = \frac{1}{g^2(\mu)} + \frac{T(A)}{16\pi^2} \log \frac{1}{g_c^2(\mu)} \tag{1.115}$$

Now it is straightforward

$$\beta\left(\frac{1}{g_c^2}\right)\left(1 - \frac{T(A)}{16\pi^2} g_c^2\right) = \beta\left(\frac{1}{g^2}\right) \tag{1.116}$$

and finally, from (1.107) obtain the $NSVZ$ beta function.

$$\boxed{\beta(g_c) = -\frac{g_c^3}{32\pi^2}\frac{3T(A) - \sum_f T(R_f)(1-2\gamma_f(g_c))}{1 - \frac{g_c^2 T(A)}{16\pi^2}}} \tag{1.117}$$

Notice that, when we write the lagrangian canonically normalized, even for pure $N = 1$ SYM, the beta function receives corrections from all loops. Expanding (1.117) in power series of g_c^2 should match all the higher loop perturbative calculations in a very particular scheme.

Discrete Chiral Symmetry.

The original flavour symmetry of the UV action(1.50) is $U(N_f) \times U(N_f) \times U(1)_R \sim SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_A \times U(1)_R$. The quantum numbers under these groups of the chiral and vector multiplets are given in table 1. In order to see how they arise remember that the chiral anomaly of a single left-handed Weyl fermion $\psi_\alpha \to e^{i\alpha}\psi_\alpha$ shifts the θ parameter by

$$\theta \to \theta - T(R)\alpha \tag{1.118}$$

We expect a total shift of the θ parameter given by $\theta \to \theta - n\alpha$ where n receives contributions from all the chiral fermions in the spectrum.

Notice that, since θ is an angular variable with period 2π, an exact discrete remnant Z_n (gauge) symmetry of any anomalous $U(1)$ is given by $\alpha = m\frac{2\pi}{n}$, $m = 1, 2, ..., n$. Let us compute n in some cases.

	$2C(R)$	$SU(N_f)_L \times SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$	$U(1)_{AF}$
θ	0	0	0	0	1	1
$Q_f \sim \phi_f$	0	$(\ N_f\ ,\ 1\ )$	1	1	0	$\frac{N_f - N_c}{N_f}$
ψ_f	1	$(\ N_f\ ,\ 1\ )$	1	1	-1	$-\frac{N_c}{N_f}$
$\tilde{Q}^f \sim \tilde{\phi}^f$	0	$(\ 1\ ,\ \bar{N}_f\ )$	-1	1	0	$\frac{N_f - N_c}{N_f}$
$\tilde{\psi}^f$	1	$(\ 1\ ,\ \bar{N}_f\ )$	-1	1	-1	$-\frac{N_c}{N_f}$
V^a	0	0	0	0	0	0
$W^a \sim \lambda^a$	$2N_c$	0	0	0	1	1
n			0	$2N_f$	$2(N_c - N_f)$	0

Table 1.

From the last column observe that $U(1)_{AF} = U(1)_R + (1 - \frac{N_c}{N_f})U(1)_A$ is the anomaly free combination of $U(1)_R$ and $U(1)_A$, as is revealed by the zero in the last entry. In summary, anomalies break $U(1)_A \hookrightarrow Z_{2N_f}$ and $U(1)_R \hookrightarrow Z_{2(N_c-N_f)}$, while there is an anomaly free combinations $U(1)_{AF}$ which remains unbroken.

1.4. Non Perturbative Superpotentials

1.4.1. $N_f < N_c$ massless: The Affleck-Dine-Seiberg Superpotential

The non perturbative induced superpotential must be a certain function of the massless degrees of freedom $\mathcal{W} = \mathcal{W}(M^f{}_g)$, constrained by the symmetries of the microscopic action. The only true (non-anomalous) symmetry $U(1)_{AF}$ is an $R-$symmetry, and hence $\mathcal{W}$ should transform with weight 2 under it. From table 1 we ee that $detM \sim \det(Q\tilde{Q})$ transforms with AF charge $2/(N_c - N_f)$. Correct scaling dimension 3 is achieved by adjoining the apropriate powers of Λ. The unique answer is given by the so called Affleck-Dine-Seiberg superpotential:

$$\boxed{\mathcal{W}_{dyn} = (N_c - N_f)\left(\frac{\Lambda^b}{\det M}\right)^{1/(N_c-N_f)}} \tag{1.119}$$

where, remember, $b = 3N_c - N_f$. In view of the extension to more complicated examples we shall review an alternative and fully equivalente route here. Forget about $U(1)_{AF}$, and turn all anomalous symmetries non-anomalous by endowing the parameters if the theory with compensating tranformation rules. In the case at hand, if θ changes as

$$\theta \to \theta + n\alpha \qquad \Rightarrow \tau \to \tau + \frac{n}{2\pi}\alpha \tag{1.120}$$

this compensates for the anomalous jacobian 1.118. This is nothing but a trick, that handles anomalous symmetries as spontaneously broken symmetries, by regarding the parameters in the accion as *vev*'s of background fields in some more fundamental theory. After integrating out the massive fields, the low energy effective action still will parametrically depend on these background fields. Now the symmetries must be respected by the low energy effective superpotential, which, as a function includes the parameters.

At low energy the dynamical scale Λ also gets transformed, through its link with the microscopic coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$.

$$\Lambda^b = \mu e^{2\pi i\tau(\mu)} = \mu e^{-\frac{8\pi^2}{g^2}+i\theta} \quad \to \quad \Lambda^b e^{in\alpha} \tag{1.121}$$

With all this we may build the following table

	$U(1)_B$	$U(1)_A$	$U(1)_R$	dimension $U(1)_s$
$M^f{}_g$	0	2	0	2
$\det M$	0	$2N_f$	0	$2N_f$
Λ^b	0	$2N_f$	$2(N_c - N_f)$	b

Table 2.

The explicit construction of the superpotential $\mathcal{W}_{eff}$ has to cope with the following global symmetries:

- $SU_L(N_f) \times SU_R(N_f)$. This enforces ${M^f}_g$ to enter through the combination $\det M$.
- $U(1)_A$ invariance constraints $\mathcal{W}_{eff}(\Lambda, \det M) = f(\Lambda^b / \det M)$.
- $U(1)_R$. Being an R symmetry, the invariant is $d^2\theta \mathcal{W}_{eff}$ hence $\mathcal{W}_{eff}$ must have charge 2. This yields

$$\mathcal{W}_{eff} = C \left(\frac{\Lambda^b}{\det M} \right)^{1/(N_c - N_f)} \tag{1.122}$$

- $U(1)_{scale}$. The invariant is now $d^4x d^2\theta \mathcal{W}_{eff}$, so $\mathcal{W}_{eff}$ must transform with weight 3. This is true for (1.119)

$$\begin{aligned} [\mathcal{W}_{eff}] &= ([\Lambda^b] - [\det M])/(N_c - N_f) \\ &= (b - 2N_f)/(N_c - N_f) = (3N_c - N_f - 2N_f)/(N_c - N_f) = 3 \end{aligned}$$

  The fact that scale invariance is automatic once $U(1)_R$ is safe is a manifestation of the anomaly multiplet structure, *i.e.* the fact that the energy momentum tensor and the R-current belong to the same supermultiplet.

$N_f = 0$ Chiral Symmetry Breaking through Gaugino Condensation

Expression (1.119) is kown as the Affleck-Dine-Seiberg superpotential. It is ill defined for $N_f \geq N_c$. In the limit $N_f \to 0$ we still may define the Affleck-Dine-Seiberg superpotential by setting $\det M = 1$, hence using $b = 3N_c$

$$\boxed{\mathcal{W}_{dyn} = N_c \, \Lambda^{\frac{b}{N_c}} = N_c \Lambda^3} \tag{1.123}$$

This dynamically generated superpotential has two consequences

- *Gaugino Condensation* Because gauginos of supersymmetric Yang Mills theories are massless and, at low energies, strongly interacting fermions, it is natural to ask whether we can reasonably expect pair condensation. This would be like quarks in QCD or like Cooper pairs in BCS theory of superconductivity. Once we have integrated out hte gluons, the effective lagrangian written in terms of effective degrees of freedom can help us in this direction

$$\begin{aligned} \langle \lambda^a \lambda_a \rangle &= \frac{1}{Z} \int [dV] \; \lambda^a \lambda_a \; e^{\frac{\tau}{16\pi} \int d^6x \; \mathrm{tr} W^2 + h.c.} \\ &= \frac{1}{Z(\tau)} \int [dV] \;\; \mathrm{tr} W^2 \big|_{\theta=0} \; e^{\left(\frac{\tau}{16\pi i} \int d^6x \; \mathrm{tr} W^2 - h.c. \right)} \\ &= \frac{1}{Z(\tau)} \, 16\pi \frac{\partial}{\partial \tau} \left(\int [dV] \; e^{\frac{\tau}{16\pi i} \int d^6x \; \mathrm{tr}\, W^2} \right) \Bigg|_{\theta=0} \end{aligned}$$

$$
\begin{aligned}
&= 16\pi \frac{\partial}{\partial \tau} \left(\log Z[\tau]\right)|_{\theta=0} \\
&= 16\pi i \frac{\partial}{\partial \tau} \int d^2\theta \; \mathcal{W}_{eff}(\tau) \\
&= 16\pi N_c \frac{\partial}{\partial \tau} \Lambda^3(\tau) \\
&= 16\pi i N_c \frac{\partial}{\partial \tau} \mu^3 e^{\frac{2\pi i \tau}{N_c}} \\
&= -32\pi^2 \, \Lambda^3
\end{aligned} \tag{1.124}
$$

- *Spontaneous Chiral Symmetry Breaking*

  The exact discrete remnant symmetry of the microscopic theory $U(1)_R \hookrightarrow Z_{2N_c}$ (see paragraph after equation (1.207)) is spontaneously broken down to Z_2 by the gaugino condensate. Indeed under $U(1)_R$, Λ^b has charge $2N_c$ (see table 2), hence Λ^{b/N_c} has charge 2,

$$
\Lambda^3 = \Lambda^{\frac{b}{N_c}} \to \Lambda^3 e^{\frac{2N_c}{N_c} i\alpha} = \Lambda^3 e^{2i\alpha} \tag{1.125}
$$

  Therefore just $\alpha = 0, \pi$ leave the vacuum invariant. All the other elements $\alpha = m/N_c \;\; (m = 1, ..., N_c - 1)$ yield vacua which are different but gauge equivalent to the one given in (1.124). In total we have N_c vacua.

$0 < N_f < N_c$ **Runaway Vacuum**

There is no supersymmetric vacuum for finite values of $M^f{}_g$. The argument goes as follows: if $\langle M^f{}_g \rangle \neq 0$ then we expect quark condensates $\langle \tilde{\psi}_f \psi^g \rangle \neq 0$ that signal confinement and spontaneous breaking of chiral symmetry. Up to here this is the same as for gluino condensation. The point however is that $\psi_{\tilde{f}} \psi^f$ is an F-term of the composite chiral operator $\tilde{Q}_f Q^g$. Hence $F_{\tilde{Q}_g Q^f} \neq 0$ signals supersymmetry breaking. Indeed[5]

$$
F_{M^f{}_g} = \frac{\partial \mathcal{W}_{eff}}{\partial M^f{}_g} = -\frac{C}{(N_c - N_f)} \left(\frac{\Lambda^b}{\det M} \right)^{1/(N_c - N_f)} M^{-1\,g}{}_f \tag{1.126}
$$

Hence the vacuum energy

$$
\begin{aligned}
E &= \sum_{f,g} |F_{M^f{}_g}|^2 + \sum_a |D^a|^2 \\
&= \frac{C^2}{(N_c - N_f)^2} \left(\frac{\Lambda^b}{\det M} \right)^{2/(N_c - N_f)} Tr(M^{-1} M^{\dagger -1})
\end{aligned} \tag{1.127}
$$

since we have chosen the $\tilde{Q}_g, Q^f$ configurations that solve the D flatness conditions (1.18). This forces a runaway behaviour $M^f{}_g \to \infty$ for a (supersymmetric) minimum at $E = 0$.

[5] use $M^{-1}\delta M = -M^{-1\;g}{}_f \delta M^f{}_g$, with $\delta = \partial_{M^f{}_g}$.

1.4.2. $N_f < N_c$ mass deformation

Consider the mass term

$$\mathcal{W}_{tree} = \sum_{fg} m^f{}_g Q^g \tilde{Q}_f \ . \tag{1.128}$$

Again, in order to look for a non-perturbative low energy superpotential we shall impose invariance of $\mathcal{W}_{tree}$ by endowing $m^f{}_g$ with the required transformation properties. In this way we complete table 2

	$U(1)_B$	$U(1)_A$	$U(1)_R$	dimension
$\det M$	0	$2N_f$	0	$2N_f$
Λ^b	0	$2N_f$	$2(N_c - N_f)$	b
$\tilde{Q}_f Q^g$	0	2	0	2
$m^f{}_g$	0	-2	2	1
$\mathrm{tr} mM$	0	0	2	3
$\det m$	0	$-2N_f$	$2N_f$	N_f

Table 3.

Now we have more freedom, and indeed the two combinations

$$a = \left(\frac{\Lambda^b}{\det M}\right)^{\frac{1}{N_c - N_f}} \quad ; \quad b = \mathrm{tr}(mM) \tag{1.129}$$

have the right symmetries and dimensions.[6] We may parametrize the solution as

$$\begin{aligned} \mathcal{W}_{eff}(M^f{}_g, \Lambda, m) &= a f(b/a) \qquad (1.130) \\ &= a(c_0 + c_1 b/a + c_2 (b/a)^2 + ...) \\ &= c_{N_c,N_f} \left(\frac{\Lambda^b}{\det M}\right)^{\frac{1}{N_c-N_f}} + c_1 \mathrm{Tr} mM + c_2 (\mathrm{Tr} mM)^2 \left(\frac{\det M}{\Lambda^b}\right)^{\frac{1}{N_c-N_f}} + ... \end{aligned}$$

In the limit $m^f{}_g \to 0$ we know the answer constains just the first term. However, taking also the weak coupling limit $g(\mu)|_{fixed\ \mu} \to \infty$ yields $\Lambda = \mu e^{-\frac{8\pi^2}{g^2 b}} \to 0, \ \ (b > 0)$. Hence the third and ongoing terms can survive in this double scaling limit. To avoid this we conclude that $c_2 = c_3 = ... = 0$, and then obtain the simple answer $\mathcal{W}_{eff} = \mathcal{W}_{dyn} + \mathcal{W}_{tree}$,

$$\boxed{\mathcal{W}_{eff}(Q, \tilde{Q}; m, \Lambda) = (N_c - N_f)\left(\frac{\Lambda^b}{\det M}\right)^{\frac{1}{N_c-N_f}} + \mathrm{Tr}\, mM} \tag{1.131}$$

[6] Other combinations like $(\det m \det M)^{1/N_f}$ seem pathological in the limit $N_f \to 0$.

Vacuum Stabilization

There is no more runaway behavior.

$$\frac{\partial \mathcal{W}_{eff}}{\partial M^g{}_f} = -\left(\frac{\Lambda^b}{\det M}\right)^{\frac{1}{N_c - N_f}} M^{-1}{}^f{}_g + m^f{}_g = 0 \tag{1.132}$$

from here

$$\begin{aligned} m^f{}_g &= \left(\frac{\Lambda^b}{\det M}\right)^{\frac{1}{N_c - N_f}} M^{-1}{}^f{}_g && (1.133) \\ \det m &= \left(\Lambda^b\right)^{\frac{N_f}{N_c - N_f}} (\det M)^{\frac{-N_c}{N_c - N_f}} \\ \det M &= \left(\Lambda^b\right)^{\frac{N_f}{N_c}} (\det m)^{-\frac{N_c - N_f}{N_c}} && (1.134) \end{aligned}$$

Inserting (1.134) back into (1.133) we arrive at

$$\boxed{\langle M^f{}_g\rangle = \left(\Lambda^b_{N_c,N_f} \det m\right)^{\frac{1}{N_c}} m^{-1}{}^f{}_g} \tag{1.135}$$

This formula displayes the expected N_c solutions through the $(1/N_c)^{th}$ root.

What is the number of vacua?. In general the flavor symmetry is broken by the mass term (1.128) as follows

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_B \times U(1)_A \hookrightarrow U(1)_B$$

unless all masses are equal $m^f{}_g = m\delta^f{}_g$ in which case also the vectorial $SU(N_f)_V$ survives. Notice that although neither $U(1)_A$ nor $U(1)_R$ are classical symmetries, there is a combination $U(1)_{\tilde{R}} = U(1)_A + U(1)_R$ that is. Actually it is equivalent to redefining the R-charge of the chiral superfields $Q, \tilde{Q} \to 1$ in Table 1 (the common value of R charge for the quiral superfields is fixed by the transformation of the superpotential, otherwise it is free). From the same table we observe that for this combination $n = 2N_c$ and this signals, that the continuous classical symmetry is broken by instantons to the discrete quantum symmetry Z_{2N_c}, as in the pure gauge theory. The elements of this discrete group are phases of the form $e^{im\frac{2\pi}{2N_c}}, m = 1, ..., 2N_c$. The chiral field condensate $M^f{}_g$ as charge 2 under this discrete remnant symmetry, hence it transforms with $e^{2mi\frac{2\pi}{2N_c}}$ which is left invariant only by the two elements $m = 0, N_c$. As in the flavourless case, this signals spontaneous symmetry breaking $Z_{2N_c} \hookrightarrow Z_2$ in as much as the truely distinct vacua are given by

$$\langle M^f{}_g\rangle = c\; e^{i(2m)\frac{2\pi}{2N_c}} \quad ; \quad m = 1, ..., N_c$$

with c as in the right hand side of (1.135). They are mapped to one another by the broken symmetry Z_{2N_c}/Z_2 given by $m = 1, ..., N_c$. Therefore we again obtain N_c vacua in complete agreement with index theorem calculations.

Gaugino Condensation in the Unbroken $SU(N_c - N_f)$

This is more or less obvious from the exponent $1/(N_c - N_f)$ in (1.119). We can see it in two different, but equivalent ways

- For light fields, *i.e.* $m^f{}_g \to 0$, we see that the vacuum condition (1.135) pushes the *v.e.v.*s $\langle M^f{}_g \rangle \to \infty$. We have here a case of matching at theory scale $\mu^{N_f} \sim \det\langle M \rangle$ between a high energy with $SU(N_c)$ theory with N_f flavours and a low energy theory pure gauge theory with $SU(N_c - N_f)$ flavours:

$$\Lambda_{N_c-N_f}^{3(N_c-N_f)} = \frac{\Lambda_{N_c}^{3N_c-N_f}}{(\det\langle M \rangle)} \tag{1.136}$$

  Dimensions match because $\det M$ has dimension $2N_f$. Indeed, (1.136) is the only gauge and flavour group invariant version of (1.79) that we have at hand. In terms of this the superpotential (1.119) reads

$$\mathcal{W} = (N_c - N_f)\left(\Lambda_{N_c-N_f}^{3(N_c-N_f)}\right)^{\frac{1}{N_c-N_f}} = (N_c - N_f)\,\Lambda_{N_c-N_f}^{3} \tag{1.137}$$

  In view of (1.124) we expect a condensate

$$\langle \lambda\lambda \rangle = 32\pi^2 \Lambda_{N_c-N_f}^3 \tag{1.138}$$

- Take now values of $m^f{}_g >> 1$ all flavours become very massive with expectation values given by (1.135). We may integrate them out, and insert them in the *Konishi anomaly* relation (1.94)

$$\langle \lambda\lambda \rangle = 32\pi^2 \left(\Lambda_{N_c,N_f}^{3N_c-N_f} \det m\right)^{\frac{1}{N_c}} \tag{1.139}$$

  Again we are in the presence of a matching of theories $(SU(N_c), N_f) \overset{\mu=m^f{}_g}{\rightarrow} (SU(N_c), 0)$ at scale $\det m$,

$$\Lambda_{N_c}^{3N_c} = \Lambda_{N_c,N_f}^{3N_c-N_f} \det m \tag{1.140}$$

  Hence we find again a gaugino condesate in the remaining pure gauge theory

$$\langle \lambda\lambda \rangle = 32\pi^2 \Lambda_{N_c}^3 \tag{1.141}$$

Holomorphic Decoupling

This is a powerfull tool to check the consistency of the Affleck Dine Seiberg superpotential. With the prefactor $N_c - N_f$ the different values of N_f chain together nicely. To see this, let the matrix $m^f{}_g$ be dominated by $m^{N_f}{}_{N_f} = m >> m^h{}_d \sim 0$. Then, instead of (1.131) and (1.132) we find

$$\mathcal{W}_{eff}(\Lambda_{N_c,N_f};m) \hookrightarrow (N_c - N_f)\left(\frac{\Lambda^b_{N_c,N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} + mM^{N_f}{}_{N_f} \tag{1.142}$$

$$\frac{\partial \mathcal{W}_{eff}}{\partial M^h{}_{N_f}} = -\left(\frac{\Lambda^b_{N_c,N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} (M^{-1})^h{}_{N_f} = 0 \tag{1.143}$$

$$\frac{\partial \mathcal{W}_{eff}}{\partial M^{N_f}{}_{N_f}} = -\left(\frac{\Lambda^b_{N_c,N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} (M^{-1})^{N_f}{}_{N_f} + m = 0 \tag{1.144}$$

From equation (1.143) we learn that $(M^{-1})^h{}_{N_f} = 0$. This means that M^{-1}, and hence M, are block diagonal matrices. Now, due to this fact we have that $(M^{-1})^{N_f}{}_{N_f} \sim 1/M^{N_f}{}_{N_f}$, as well as $\det M \sim \det \tilde{M} \cdot M^{N_f}{}_{N_f}$ and, hence, from (1.144) we obtain

$$M^{N_f}{}_{N_f} = \left(\left(\frac{\Lambda^{3N_c-N_f}_{N_c,N_f}}{\det \tilde{M}}\right)^{\frac{1}{N_c-N_f}} \frac{1}{m}\right)^{\frac{N_c-N_f}{N_c-N_f+1}} \tag{1.145}$$

Inserting this back into (1.142) we get after a little algebra

$$\mathcal{W}_{eff} = (N_c - N_f + 1)\left(\frac{m\,\Lambda^{3N_c-N_f}_{N_c,N_f}}{\det \tilde{M}}\right)^{\frac{1}{N_c-N_f+1}} \tag{1.146}$$

Invoking now the matching condition (1.79) with $\tilde{N}_c = N_c$ and $\tilde{N}_f = N_f - 1$ gives

$$m\Lambda^{3N_c-N_f}_{N_f,N_c} = \Lambda^{3N_c-N_f+1}_{N_c,N_f-1} \tag{1.147}$$

in the limit $m \to \infty$ we obtain the correct decoupling, including the prefactor

$$\mathcal{W}_{eff}(\Lambda_{N_c,N_f};m) \overset{m\to\infty}{\hookrightarrow} \mathcal{W}_{eff}(\Lambda_{N_c,N_f-1}) = (N_c - N_f + 1)\left(\frac{\Lambda^{3N_c-N_f-1}_{N_c,N_f-1}}{\det \tilde{M}}\right)^{\frac{1}{N_c-N_f+1}} \tag{1.148}$$

1.4.3. Integrating Out and In

General Strategy for Nonperturbative Effective Superpotentials

We will now try to put the previous exercise in a more general framework. The general setup deals with a supersymmetric field theory and a treel level superpotential

$$\mathcal{W}_{tree} = \sum_r g_r X^r(\Phi^i) \tag{1.149}$$

where g_r are classical sources for the gauge invariant functions X_r of the chiral superfields $\Phi^i = \phi^i + ...$ transforming in the representation R^i of the gauge group $G = \prod_s G_s$, where each factor has an associated dynamical scale Λ^{b_s}. The general procedure will be as follows

1. Set $\mathcal{W}_{tree}=0$. We have a *classical moduli space* where $V=\sum_s |D^a|^2=0$, The *vev*'s $\langle\phi^i\rangle\neq 0$ break part or the full gauge symmetry. The gauge invariant functions X_r are natural coordinates that label inequivalent vaua. They are light fields in that the classical superpotential for them vanishes. If the X_r are constained classically we can add a lagrange multiplier to the effective action.

2. Turn on g_r and Λ (*i.e.* consider the full quantum theory. The full quantum wilsonian superpotential for the effective theory is constrained by two *kinematic constraints*

   - *Holomorphy:* $\mathcal{W}_{eff}$ is a holomorphic function of the fields X^r and the couplingsg_r. This last can be justified by thinking ofthe couplings as background *vev*'s of other superfields.
   - *Symmetries:* $\mathcal{W}_{eff}$ is constrained by *all the classical symmetries* of the theory with $\mathcal{W}_{tree}=0$. Some of these symmetries can become anomalous for $\Lambda\neq 0$, or explicitely broken by $\mathcal{W}_{tree}$. The selection rules are obtained by giving to Λ and g_r compensating transformation rules. With them, an ansatz for $\mathcal{W}_{eff}$ can be constructed. See for example (1.130) .

3. Explore the asymptotic behaviour. This includes *weak coupling limit* $g_r,\Lambda\to 0$, and large *vev*'s $\langle\phi^i\rangle >> 0$. The key fact is that a holomorphic function, $\mathcal{W}_{eff}$ in thi case, is determined by the asymptotic behavour and its singularities.

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At the end of the day, *it is often the case that the resulting answer* is of the form

$$\boxed{\begin{aligned}\mathcal{W}_{eff}(\Lambda,g_r;X^r) &= \mathcal{W}_{eff}(g_r\!=\!0)+\textstyle\sum_r g_rX^r\\ &= \mathcal{W}_{dyn}(\Lambda;X^r)+\mathcal{W}_{tree}(g_r;X^r)\end{aligned}} \qquad (1.150)$$

In the word, *it is linear in the sources* g_r. This is the case for the example (1.130). It is also the case in most of the examples known. When it is not, it is conjectured that there is a redefinition of X^r as funtions of g_r such that (1.150) holds. This is the famous *linearity principle.*

Integrating Out

Select one of the chiral fields $\hat\Phi^i$ in the original theory, and let $X_{\hat r}$ denote a subset of gauge invariant operators made out of $\hat\Phi$. Among them there is for example a mass term $g_{\hat r}X^{\hat r}=\hat m\tilde{\hat\Phi}\hat\Phi$, or higher order invariants.

$$\mathcal{W}_{eff}(\Lambda,g_r,g_{\hat r};X^r,X^{\hat r})=\mathcal{W}_{dyn}(\Lambda;X^r)+\sum_r g_rX^r+\sum_{\hat r} g_{\hat r}X^{\hat r} \qquad (1.151)$$

We may integrate out $X^{\hat r}$ by solving their equations of motion

$$\left.\frac{d\mathcal{W}_{eff}}{dX^{\hat r}}\right|_{X^{\hat r}=\langle X^{\hat r}\rangle}=0 \qquad \hat r=1,2,... \qquad (1.152)$$

This is the same as demanding

$$\boxed{\frac{\partial \mathcal{W}_{dyn}(\Lambda; X^r)}{\partial X^{\hat{r}}} = -g_{\hat{r}}} \tag{1.153}$$

From here we obtain $\langle X^{\hat{r}} \rangle = \langle X^{\hat{r}} \rangle(\Lambda, g_r, g_{\hat{r}}, X^r)$, and inserting back into $\mathcal{W}_{eff}$ we find the effective potential for the low energy theory below scale $\hat{m}$.

$$\tilde{\mathcal{W}}_{eff}(\Lambda, g_r, g_{\hat{r}}; X^r) = \mathcal{W}_{eff}(\Lambda, g_r, g_{\hat{r}}; X^r, \langle X^{\hat{r}} \rangle) \tag{1.154}$$

The "down"theory is no more linear in the couplings $g_{\hat{r}}$. However it is still linear in the g_r

$$\frac{d\tilde{\mathcal{W}}_{eff}}{dg_r} = \frac{\partial \mathcal{W}_{eff}(\langle X^{\hat{r}} \rangle)}{\partial g_r} \; + \; \frac{\partial \langle X^{\hat{r}} \rangle}{\partial g_r} \frac{\partial \mathcal{W}_{eff}(\langle X^{\hat{r}} \rangle)}{\partial X^{\hat{r}}} = X^r \tag{1.155}$$

Therefore we write for it

$$\tilde{\mathcal{W}}_{eff}(\Lambda, g_r, g_{\hat{r}}; X^r) = \tilde{\mathcal{W}}_{dyn}(\Lambda, g_{\hat{r}}; X^r) + \sum_r g_r X^r \tag{1.156}$$

The effective dynamical potential, contains now the additional couplings $g_{\hat{r}}$. The asymptotic behaviour dictates that

$$\tilde{\mathcal{W}}_{dyn}(\Lambda, g_{\hat{r}}; X^r) = \tilde{\mathcal{W}}^0_{dyn}(\tilde{\Lambda}; X^r) + \tilde{\mathcal{W}}_I(\Lambda, g_{\hat{r}}; X^{\hat{r}}) \; . \tag{1.157}$$

where $\tilde{\mathcal{W}}_{dyn}(\Lambda; X^r)$ is the dynamical potential of the "down"theory. The term $\tilde{\mathcal{W}}_I$ vanishes in the limit $\hat{m}/g_{\hat{r}} \to \infty$. This can be understood from the fact that the perturbation $g_{\hat{r}} = \hat{m}$ is gaussian, and hence, it can be integrated out exactly in the microscopic theory. The net effect can be absorbed into a redefinition of the dynamical scale through the usual matching condition

$$\tilde{\Lambda}^{\tilde{b}} = \Lambda^b \hat{m}^{\tilde{b}-b}$$

This is fully accounted for by the first term $\tilde{\mathcal{W}}^0_{dyn}$ in (1.157).

Integrating In

Although we got rid of the variables $X_{\hat{r}}$, we actually did not lose any information. This is very different from the usual idea of integrating degrees of freedom, and is related to the linearity principle. In fact, $\tilde{\mathcal{W}}$, *is simply a Legendre transform of* $\mathcal{W}$ *with respect to the variables* $X_{\hat{r}}$. Therefore, in principle, it is possible to go back, and rescue the original variables by means of an inverse Legendre transform. To be precise, consider starting from the "down"theory $\mathcal{W}_{dyn}$ and perturbing by adding gauge singlets $X^{\hat{r}}$ at tree level. By the linearity principle, the full effective action will look as follows

$$\tilde{\mathcal{W}}_{eff}(\tilde{\Lambda}, g_r, g_{\hat{r}}; X^r, X^{\hat{r}}) = \mathcal{W}_{dyn}(\tilde{\Lambda}, g_{\hat{r}}; X^r) + \sum_r g_r X^r - \sum_{\hat{r}} g_{\hat{r}} X^{\hat{r}} \tag{1.158}$$

As usual, in this context we integrate out $g_{\hat{r}}$ by demanding independence of the l.h.s.

$$\left.\frac{d\tilde{\mathcal{W}}_{eff}}{dg_{\hat{r}}}\right|_{\langle g_{\hat{r}} \rangle} = 0 \tag{1.159}$$

This equate to

$$\boxed{\frac{\partial \tilde{\mathcal{W}}_{dyn}}{\partial g_{\hat{r}}} = X^{\hat{r}}} \tag{1.160}$$

This equation is to be solved for $\langle g_{\hat{r}} \rangle = \langle g_{\hat{r}} \rangle(\Lambda, g_r, X^r, X^{\hat{r}})$ one can show that the resulting expression is linear in g_r (and, of course, independent of $g_{\hat{r}}$). By consistency it can be no other than the original perturbed action for the "up"theory

$$\mathcal{W}_{eff}(\tilde{\Lambda}, g_r, \langle g_{\hat{r}} \rangle; X^r, X^{\hat{r}}) = \mathcal{W}_{dyn}(\Lambda; X^r, X^{\hat{r}}) \; + \; \sum_r g_r X^r \tag{1.161}$$

Veneziano-Yankielowicz Glueball Effective Superpotential

In all the previous reasonings, Λ^b appeared as a parameter that, in principle coul be treated o equal footing with the g_r. hence we may consider $\tau = \log \Lambda^b$ as the source for the composite gauge invariant chiarl superfield $S = \frac{1}{16\pi^2}\mathrm{tr} W^2$. The effective ADS potential $\tilde{\mathcal{W}} = N_c \Lambda^3 = N_c \Lambda^{\frac{b}{N_c}}$ serves to the purpose of computing the associated classical field

$$\langle S \rangle = \frac{\partial \tilde{\mathcal{W}}_{dyn}(\Lambda^b)}{\partial \log \Lambda^b} = \Lambda^{\frac{b}{N_c}} = \Lambda^3 \tag{1.162}$$

If we integrate this field in, we are entitled to solve for $\langle \log \Lambda^b(S) \rangle = N_c \log S$ and introduce it back to get

$$\begin{aligned} \mathcal{W}_{eff}(S) &= \tilde{\mathcal{W}}_{dyn}(\langle \log \Lambda^b \rangle) - S \langle \log \Lambda^b \rangle \\ &= N_c S - S \log S^{N_C} \\ &= N_c S (1 - \log S) \end{aligned}$$

From here the effective action follows as

$$\begin{aligned} \mathcal{W}_{eff}(S; \Lambda^b) &= \mathcal{W}_{dyn}(S) + \log \Lambda^b S \\ &= S \left[\log \left(\frac{\Lambda^b}{S^{N_c}} \right) + N_c \right] \\ &= N_c S \left[1 - \log \left(\frac{S}{\Lambda^3} \right) \right] \end{aligned} \tag{1.163}$$

However the meaning of this superpotential is unclear, since the chiral superfield S is massive and our effective actions are Wilsonian.

One could further integrate N_f flavours in. The procedure is now clear: add $\mathcal{W}_{tree} = \mathrm{Tr} m M$ and integrate out the parameters ${m^f}_g$. Of course, to go to the "up"theory the scales have to match. Calling $\tilde{\Lambda}^{\tilde{b}}$ th scale-coupling that appears in (1.163), and Λ^b the one of the "upstairs"theory, we have clearly

$$\tilde{\Lambda}^{\tilde{b}} = \tilde{\Lambda}^{3N-c-N_f} \prod_{m_f}^{N_f} = \Lambda^b \det m. \tag{1.164}$$

So, finally

$$\mathcal{W}_{eff}(\Lambda, m^f{}_g; S, M^f{}_g) = S\left[\log\left(\frac{\Lambda^b \det m}{S^{N_c}}\right) + N_c\right] - \mathrm{Tr} m M \qquad (1.165)$$

Solving for $d\mathcal{W}_{eff}/dm = 0$ yields

$$\langle m\rangle^f{}_g = S M^{-1\,f}{}_g \quad ; \quad \det\langle m\rangle = \frac{S^{N_f}}{\det M} \qquad (1.166)$$

and hence we obtain for the "upstairs"dynamical potential

$$\mathcal{W}_{dyn}(\Lambda; S, M^f{}_g) = S\left[\log\left(\frac{\Lambda^b}{S^{N_c-N_f}\det M}\right) + N_c - N_f\right] \qquad (1.167)$$

Because S is massive, it should be integrated out. After doing so, the Affleck-Dine-Seiberg dynamical superpotential (1.119) is recovered.

1.4.4. $N_f \geq N_c$: The quantum moduli space

No Dynamically Generated of Superpotential

When $N_c = N_f$ the $U(1)_R$ is anomaly free, *ie* Λ^b is inert. This makes it imposible to build a charge 2 superpotential in terms of Λ (at least for massless $Q, \tilde{Q}$). For $N_f \geq N_c$ the same expression as (1.119) could be used. However, the exponent $1/(N_c - N_f)$ now is negative, and hence the dynamically generated scale Λ^b appears in the denominator. This makes it impossible to arise as an expression generated by instantons. As a consequence, contrarily to the case $N_f < N_c$, for $N_f \geq N_c$ nondynamical superpotential is generated and, therefore, a moduli space is still present in the quantum theory.

Recall that the classical moduli space admits gauge invariant coordinates

$$M^f{}_g = Q^f \tilde{Q}_g$$

$$B^{[f_1 \cdots f_{N_c}]} = Q^{f_1 i_1} Q^{f_2 i_2} \cdots Q^{f_{N_c} i_{N_c}} \; \epsilon_{i_1 i_2 \cdots i_{N_c}} \qquad (1.168)$$

$$\tilde{B}_{[f_1 \cdots f_{N_f}]} = \tilde{Q}_{f_1 i_1} \tilde{Q}_{f_2 i_2} \cdots \tilde{Q}_{f_{N_c} i_{N_c}} \; \epsilon^{i_1 i_2 \cdots i_{N_c}} \qquad (1.169)$$

subject to algebraic constraints which are trivially satisfied in terms of the microscopic fields Q and $\tilde{Q}$. For $N_c = N_f$ the quantum moduli space is a deformation of the classical one. For $N_c = N_f + 1$ quantum and classical moduli spaces are the same.

$N_c = N_f$: Quantum Deformed Moduli Space

In this case there are $N_f + 2$ gauge invariant chiral composite superfields $M^f{}_g$; B and $\tilde{B}$. The constraint $f(M, B, \tilde{B}) = \det M - B\tilde{B} = 0$ leads to a singular manifold at $B = \tilde{B} = 0$, where $f = df = 0$. Seiberg proposed that quantum corrections modify this constraint in the following form

$$\boxed{\det M - B\tilde{B} = \Lambda^{2N_c}} \qquad (1.170)$$

The right hand side, being Λ^b is proportional to a one-instanton contribution. Observe that, now, the origin $M^f{}_g = B = \tilde{B} = 0$ is no more in the moduli space. Hence quantum corrections have smoothed out the singularity.

As a check of this conjecture we may decouple one flavour and try to recover the known result for $N_f = N_c - 1$. To this aim, add the tree level mass term for the last flavour $(Q^{N_f}, \tilde{Q}_{N_f})$

$$\mathcal{W}_{tree} = mQ^{N_f}\tilde{Q}_{N_f} = mM^{N_f}{}_{N_f} = \mathcal{W}_{eff} \tag{1.171}$$

since no additional piece is generated dynamically. For large m, the meson field $t = M^{N_f}{}_{N_f}$ looses all its excitations (the kinetic term can be dropped) and it should be replaced by a function of the other fields that solves de quantum constraint. Writing

$$\det M = \sum_{g=1}^{N_f} (-1)^{N_f+g} M^{N_f}{}_g (\det \tilde{M}^g{}_{N_f}) \tag{1.172}$$

where by $(\det \tilde{M}^g{}_{N_f})$ we mean the minor of the matrix element $M^{N_f}{}_g$. Hence from the constraint we can solve for $M^{N_f}{}_{N_f}$ and insert into (1.171) to find

$$\mathcal{W}_{eff} = \frac{m}{(\det \tilde{M})} \left(\Lambda^{2N_c} + B\tilde{B} - \sum_{g=1}^{N_f-1} (-1)^{N_f+g} M^{N_f}{}_g (\det\tilde{M})^g{}_{N_f} \right) \tag{1.173}$$

with $(\det \tilde{M}) = (\det \tilde{M}^{N_f}{}_{N_f})$. To find the vacuum manifold, we must solve the F flatness conditions. In particular for $f, g = 1, ..., N_f - 1$ we find

$$\begin{aligned}
\frac{\partial \mathcal{W}_{eff}}{\partial M^f{}_h} &= \sum_{g=1}^{N_f-1} m(-1)^{N_f+g} M^{N_f}{}_g \frac{\partial}{\partial M^f{}_h} \left(\frac{(\det \tilde{M})^g{}_{N_f}}{(\det \tilde{M})} \right) = 0 \\
\frac{\partial \mathcal{W}_{eff}}{\partial B} &= \frac{\tilde{B}}{(\det \tilde{M})} = 0 \\
\frac{\partial \mathcal{W}_{eff}}{\partial \tilde{B}} &= \frac{B}{(\det \tilde{M})} = 0
\end{aligned} \tag{1.174}$$

which is solved generically with $M^{N_f}{}_g = B = \tilde{B} = 0$. Inserting this back into (1.173) only the first term survives

$$\begin{aligned}
\mathcal{W}_{eff} &= \frac{m\Lambda^{2N_c}}{\det \tilde{M}} = \left.\frac{m\Lambda^{3N_c-N_f}}{\det \tilde{M}}\right|_{N_f=N_c} \\
&= \left.\frac{m\Lambda^b_{N_c,N_f}}{\det \tilde{M}}\right|_{N_f=N_c} = \left.\frac{\Lambda^{\tilde{b}}_{N_c,\tilde{N}_f}}{\det \tilde{M}}\right|_{\tilde{N}_f=N_c-1} \\
&= (N_c - \tilde{N}_f) \left.\frac{\Lambda^{\tilde{b}}_{N_c,\tilde{N}_f}}{\det \tilde{M}}\right|_{\tilde{N}_f=N_c-1}
\end{aligned}$$

hence the *down* superpotential comes out with the correct coefficient.

The constrained manifold allows for different patterns of symmetry breaking depending on the *vev*'s of M, B and $\tilde{B}$. However in no point of moduli space is the full chiral symmetry broken. Indeed, for $N_c = N_f$ and from table 1, we recognize that al scalar quarks are neutral under $U(1)_{AF}$, hence this group will generically remain unbroken. Other points of interest show partial breaking of the global symmetry are

- $M^f{}_g = \Lambda^2 \delta^f{}_g$, $B = \tilde{B} = 0$. At this point we have a pattern of symmetry breaking $SU(N_f)_L \times SU(N_f)_R \times U(1)_b \times U(1)_{AF} \to SU(N_f)_V \times U(1)_B \times U(1)_{AF}$
- $M^f{}_g = 0$, $B = -\tilde{B} = \Lambda^{N_c}$, where the pattern of symmetry breaking now is $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_{AF} \to SU(N_f)_L \times SU(N_f)_R \times U(1)_{AF}$

At either of these points we have a certain current algebra realized in terms of two theories, a microscopic $(V, Q, \tilde{Q})$ and a macroscopic $(M, B, \tilde{B})$. A strong test of the picture is provided by anomalous triangles, which can be computed in either of both theories, and must yield the same answer, as is indeed the case.

$N_f = N_c + 1$

In this situation we have $N_f^2 + 2N_f$ gauge invariant polynomials $M^f{}_g, B^f$ and $\tilde{B}_f$ satisfying the classical constraints

$$\bar{B}_f M^f{}_g = M^f{}_g \bar{\tilde{B}}^g = 0 \quad ; \quad \det M (M^{-1})_f{}^g = \bar{B}_f \bar{\tilde{B}}^g \tag{1.175}$$

where

$$\bar{B}_f = \epsilon_{f f_1 \dots f_{N_c}} B^{[f_1 \dots f_{N_c}]} \quad ; \quad \bar{\tilde{B}}_f = \epsilon^{f f_1 \dots f_{N_c}} \tilde{B}_{[f_1 \dots f_{N_c}]} \; ; \tag{1.176}$$

We cannot deform this constraint in a way covariant with the global symmetries in the absence of masses. Therefore the classical constraints remain the same. In fact the (1.35) can be thought of as the vacuum equations $\partial\mathcal{W}/\partial M = \partial\mathcal{W}/\partial B = \partial\mathcal{W}/\partial\tilde{B} = 0$ coming from the following auxiliary superpotential

$$\mathcal{W} = \frac{1}{\Lambda^b} (\bar{\tilde{B}}_g M^g{}_f \bar{B}^f - \det M) \tag{1.177}$$

This should not be thought of as a dynamically generated superpotential, but rather as a trick to envisage the constrained manifold (1.175) as the vacuum surface embedded in a larger space where all fields $M^f{}_g, B$ and $\tilde{B}$ are physical and independent but couple through (1.177).

As a check of this ansatz, let us decouple the last flavour, by adding a mass term to (1.177)

$$\mathcal{W} = \frac{1}{\Lambda^b} (\bar{\tilde{B}}_g M^g{}_f \bar{B}^f - \det M) - m M^{N_f}{}_{N_f} \tag{1.178}$$

The F flatness equations $\partial\mathcal{W}/\partial M^{N_f}{}_i = \partial\mathcal{W}/\partial M^i{}_{N_f} = 0$ with $i < N_f$ reduce M to the following form

$$M = \begin{pmatrix} \tilde{M} & 0 \\ 0 & M^{N_f}{}_{N_f} \end{pmatrix} \quad ; \quad \bar{B}_f = \begin{pmatrix} 0 \\ \bar{B}_{N_f} \end{pmatrix} \quad ; \quad \bar{\tilde{B}}^f = \begin{pmatrix} 0 \\ \bar{\tilde{B}}^{N_f} \end{pmatrix} \tag{1.179}$$

so, renaming $\bar{B}_{N_f} = B$ and $\bar{\tilde{B}}_{N_f} = \tilde{B}$, the superpotential takes the form

$$\mathcal{W} = \left((\frac{1}{\Lambda^b} \tilde{B} B - \det \tilde{M}) - m \right) M^{N_f}{}_{N_f} \tag{1.180}$$

Finally the equation of motion $\partial \mathcal{W} / \partial M^{N_f}{}_{N_f} = 0$ implies, with $\Lambda^b = \Lambda^{2N_c - 1}_{N_c, N_c+1}$

$$\det \tilde{M} - B\tilde{B} = m\Lambda^{2N_c-1}_{N_c,N_c+1} = \Lambda^{2N_c}_{N_c,N_c} \tag{1.181}$$

and the correcto deformed moduli space for $N_c = N_f$ is recovered.

The point $M = B = \tilde{B} = 0$ is now in the vacuum manifold. At this point the full global symmetry of the lagrangian is preserved. Hence we have confinement without chiral symmetry breaking. At this point, anomalies provides a maximal set of nontrivial triangles to be matched.

$N_f \geq N_c + 2$

The previous construction does not admit a further generalization to higher values of $N_f - N - c \geq 2$. There is no way to write a superpotential with $U(1)_{AF}$ charge 2 that reproduces the classical constraints through the vacuum equations, and the anomaly matching conditions are not satisfied. A better idea is needed.

1.5. Seiberg Duality

Recall the original form of the barionic operators given in (1.27) and (1.28) and repeated below

$$\begin{aligned} B^{[f_1 \dots f_{N_c}]} &= Q^{f_1 i_1} \dots Q^{f_{N_c} i_{N_c}} \epsilon_{i_1 \dots i_{N_c}} & (1.182) \\ \tilde{B}_{[f_1 \dots f_{N_c}]} &= \tilde{Q}_{f_1 i_1} \dots \tilde{Q}_{f_{N_c} i_{N_c}} \epsilon^{i_1 \dots i_{N_c}} & (1.183) \end{aligned}$$

The antisymmetric set of indices $N_c \leq N_f$ indices signals that this is a bound state of N_c objects (quarks $(Q^i, \tilde{Q}_i)$) transforming in the fundamental of $SU(N_c)$ gauge. Seiberg's proposal starts by going over to the Hodge duals

$$\bar{B}_{[g_1 \dots g_{\tilde{N}_c}]} = \frac{1}{N_c!} \epsilon_{g_1 \cdots g_{\tilde{N}_c} f_1 \cdots f_{N_c}} Q^{f_1 i_1} Q^{f_2 i_2} \cdots Q^{f_{N_c} i_{N_c}} \epsilon_{i_1 i_2 \cdots i_{N_c}} \tag{1.184}$$

$$\bar{\tilde{B}}^{[g_1 \dots g_{\tilde{N}_c}]} = \frac{1}{N_c!} \epsilon^{g_1 \cdots g_{\tilde{N}_c} f_1 \cdots f_{N_c}} \tilde{Q}_{f_1 i_1} \tilde{Q}_{f_2 i_2} \cdots \tilde{Q}_{f_{N_c} i_{N_c}} \epsilon^{i_1 i_2 \cdots i_{N_c}} \tag{1.185}$$

with $\tilde{N}_c = N_f - N_c$ and interpret them as bound states of $\tilde{N}_c$ objects (dual quarks $(q_i, \tilde{q}^i)$) transforming in the fundamental of $SU(\tilde{N}_c)$.

$$\bar{B}_{[g_1 \dots g_{\tilde{N}_c}]} = q_{g_1 i_1} \cdots q_{g_{\tilde{N}_c} i_{\tilde{N}_c}} \epsilon^{i_1 i_2 \cdots i_{\tilde{N}_c}} \tag{1.186}$$

$$\bar{\tilde{B}}^{[g_1 \dots g_{\tilde{N}_c}]} = q^{g_1 i_1} \cdots q^{g_{\tilde{N}_c} i_{\tilde{N}_c}} \epsilon_{i_1 i_2 \cdots i_{\tilde{N}_c}} \tag{1.187}$$

The number of flavours is the same $N_f \geq \tilde{N}_c$.

	$SU(N_f)_L \times SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_{AF}$	$D(g=0)$	$D(g=g^*)$
Q_f	$(N_f \ , \ 1)$	1	1	$\frac{\tilde{N}_c}{N_f}$	1	$\frac{3}{2}\frac{\tilde{N}_c}{N_f}$
$\tilde{Q}^f$	$(1 \ , \ \bar{N}_f)$	-1	1	$\frac{\tilde{N}_c}{N_f}$	1	$\frac{3}{2}\frac{\tilde{N}_c}{N_f}$
${M^f}_g$	$(N_f, \bar{N}_f)$	0	2	$2\frac{\tilde{N}_c}{N_f}$	2	$3\frac{\tilde{N}_c}{N_f}$
$B^{f_1 \dots f_{\tilde{N}_c}}$	$\begin{pmatrix} N_f \\ N_c \end{pmatrix}$	N_c	N_c	$\frac{N_c \tilde{N}_c}{N_f}$	N_c	$\frac{3}{2}\frac{N_c \tilde{N}_c}{N_f}$
$\tilde{B}_{f_1 \dots f_{\tilde{N}_c}}$	$\begin{pmatrix} N_f \\ N_c \end{pmatrix}$	$-N_c$	N_c	$\frac{N_c \tilde{N}_c}{N_f}$	N_c	$\frac{3}{2}\frac{N_c \tilde{N}_c}{N_f}$

Table 3.

An important obervation is the following: gauge symmetries are related to a redundant description of physics. Duality, in the sense of universality says there is nothing that prevents having two such different UV descriptions that yield equivalent IR physics. However, global symmetries survive intact and must be the same all along the RG trajectory. The first thing we would try to do is to assign correct global charges to the dual microscopic fields. Since $U(1)$ charges are additive, the correct way to assign quantum numbers to the dual quarks $q_f, \tilde{q}^g$ is to partition those of the baryon operators, among the new constituent fields. This leads to the first two lines in the following table.

	$SU(N_f)_L \times SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_{AF}$	$D(g=0)$	$D(g=g^*)$
q_f	$(\bar{N}_f, \ 1)$	$\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{N_f}$	1	$\frac{3}{2}\frac{N_c}{N_f}$
$\tilde{q}^f$	$(1 \ , N_f)$	$-\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{\tilde{N}_c}$	$\frac{N_c}{N_f}$	1	$\frac{3}{2}\frac{N_c}{N_f}$
${T^f}_g$	$(N_f, \bar{N}_f)$	0	2	$2\frac{\tilde{N}_c}{N_f}$	1	$3\frac{\tilde{N}_c}{N_f}$
${\tilde{M}^g}_f$	$(N_f, \bar{N}_f)$	0	$2\frac{N_c}{\tilde{N}_c}$	$2\frac{N_c}{N_f}$	2	$3\frac{N_c}{N_f}$
$\bar{B}_{g_1 \dots g_{\tilde{N}_c}}$	$\begin{pmatrix} N_f \\ N_c \end{pmatrix}$	N_c	N_c	$\frac{N_c \tilde{N}_c}{N_f}$	N_c	$\frac{3}{2}\frac{N_c \tilde{N}_c}{N_f}$
$\bar{\tilde{B}}^{g_1 \dots g_{\tilde{N}_c}}$	$\begin{pmatrix} N_f \\ N_c \end{pmatrix}$	$-N_c$	N_c	$\frac{N_c \tilde{N}_c}{N_f}$	N_c	$\frac{3}{2}\frac{N_c \tilde{N}_c}{N_f}$

Table 4.

It is fairly obvious from this table that the "magnetic"meson operator ${\tilde{M}^f}_g = \tilde{q}^f q_g$ is not the counterpart of the "electric"meson operator M^f_g. For this reason, Seiberg introduced an additional fundamental field, a color singlet $T^f{}_g$ transforming in the $(N_f, \bar{N}_f)$ representation, and endowed with the same quantum charges as the "electric"meson operator $M^f{}_g$. Now, if $T^f{}_g$ is a new fundamental chiral superfield, there should be a $U(1)_T$ under wich only T is charged. This symmetry was not observed in the electric theory. Both problems, the absence of magnetic mesons $\tilde{M}^f{}_g = \tilde{q}^f q_g$ and $U(1)_T$ are solved by postulating the existence of a relevant superpotential

$$\mathcal{W}(q, \tilde{q}, T) = \lambda\, q_{fi}\, T^f{}_g\, \tilde{q}^{gi} = \lambda \mathrm{tr} T\tilde{M} \tag{1.188}$$

where λ is a dimensionless constant. Observe that, interestingly enough, this expression has the right $U(1)_B \times U(1)_{AF}$ charge $(0, 2)$ of a superpotential. However $U(1)_T$ is explicitely broken as desired. Moreover, at a generic point in moduli space $\langle T^f{}_g \rangle \neq 0$, this potential gives mass to the magnetic meson operator, which hence decouples from the low energy effective action. Next we face the following questions

1. In which sense is this duality a symmetry?
2. What are the consistency checks that it passes?

To answer the first question we start looking carefully at the RG behaviour of the dual pair. Recall that the exact beta function was given by

$$\begin{aligned} \beta(g) &= \frac{g^3}{16\pi^2} \frac{3N_c - N_f(1 - \gamma(g^2))}{1 - N_c \frac{g^2}{8\pi^2}} \\ \gamma(g^2) &= -\frac{g^2}{8\pi^2} \frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^2) \end{aligned} \tag{1.189}$$

where γ is the anomalous dimension of the mass. The structure of the numerator in (1.189) reveals a competition between the anti-screening of gluon, and the screening of matter fields. For different values of N_f and N_c, there is winning of either effect, or even an interesting window (conformal window) where they equilibrate. A carefull analysis of this exact beta function leads to the following phase diagram.

$3N_c \leq N_f$ *Non Abelian IR Free Electric Phase.*

In the last interval, $3N_c \leq N_f$, the theory loses asymptotic freedom and gains, instead, IR freedom. The potential among distant charges behaves at long distances as $V(r) \sim e^2(r)/r$ with $e^2(r) = 1/\log(\Lambda r)$. Much like QED, but with gluons (therefore the name of this phase). $r = \Lambda^{-1}$ signals, instead, a Landau pole.

$\frac{3}{2}N_c < N_f < 3N_c$ Interacting Non-Abelian Coulomb Phase, (conformal window).

In this interval, the coupling is presumed to approach a non trivial IR fixed point $g \to g^*$. This can be proven rigurously in a certain scaling limit where $N_c, N_f \to \infty$. In general it is postulated that this is always the case in this interval. Therefore, the LEEA is a nontrivial superconformal field theory. Because of conformal invariance, the potential among distant sources has to behave as $V(r) \sim 1/r$ for large r. That is why we refere to this as a non-Abelian Coulomb phase.

The full scaling dimension of a (composite) operator $D(\mathcal{O}) = \mathcal{D}_{\prime}(\mathcal{O}) + \gamma(\mathcal{O})$ *where* D_0 *is the engineering dimension and* γ *the anomalous dimension. It is to be calculated in perturbation theory as a series expansion* $\gamma = \gamma_1 g^2 + \ldots$*. Therefore the engineering dimension coincides with the full dimension in a free theory* $D_0 = D(g = 0)$, i.e. *at a trivial fixed point. The other interesting posibility arises in a superconformal field theory (for example, as is now the case, at a nontrivial fixed point* $g = g^*$*). The superconformal algebra has a nonanomalous* $U(1)_R$*. From this algebra, the exact scaling dimension of chiral operators satisfies*

$$D = \frac{3}{2}|R| \tag{1.190}$$

Back to our case, we identify $U(1)_R$ *with our anomaly free combination* $U(1)_{AF}$*. This allows us to complete the two final columns in figures 3 and 4. It is noteworthy to remark that the same values of* γ *can be obtained by demanding the vanishing of the numerator in (1.189). For example, since* Q *and* $\tilde{Q}$ *build up a single flavour* $\beta = 0 \Rightarrow \gamma(Q, \tilde{Q}) = -3\frac{N_c}{N_f} + 1$*, and hence* $D(M) = 2 + \gamma = 3\frac{\tilde{N}_c}{N_f}$*, and* $D(B) = N_c + \frac{N_c}{2}\gamma = \frac{3}{2}\frac{\tilde{N}_c N_c}{N_f}$.

$N_c + 1 < N_f \leq \frac{3}{2}N_c$ Nonabelian IR Free Magnetic Phase.

As the N_f *decreases past* $3N_c$*, the fixed point* g^* *increases. For* $N_f \leq \frac{3}{2}N_c$ *is reached* $\tilde{N} \leq N_c/2$ *and the dimension* $D(M) = \frac{3\tilde{N}_c}{N_f} \leq 1$*. This cannot correspond to a unitary superconformal field theory, for which the smalles conformal dimension is* 1*. Therefore the theory must be in a different phase. From the point of view of the electric theory looks as standard asymptotically free theory, strongly coupled in the IR. However, from the point of view of the, as yet conjectured, dual magnetic theory* $N_c + 1 < N_f \leq \frac{3}{2}N_c$ *corresponds to the inequality* $4 + \tilde{N}_c \leq 3\tilde{N}_c \leq N_f$*. Therefore the magnetic theory is in a Non-Abelian IR Free Phase. The field content is therefore best analized in the IR trivial fixed point, and consists of quarks* $q_f, \tilde{q}^f$ *and the additional field* ${T^f}_g$*. Still we have to think about the Yukawa type superpotential (1.188). The dimension of this operator, which is* 3 *at the IR trivial fixed point where* T, q *and* $\tilde{q}$ *are free, has to be computed in perturbation theory. It turns out to be irrelevant and therefore vanishing in the IR limit. Therefore the magnetic theory looks around energy scales* $\mu \to 0$ *pretty much like the electric theory around energy scales* $\mu \to \infty$.

1.5.1. Duality in the Conformal Window

Under $N_c \to \tilde{N}_c = N_f - N_c$, the band $\frac{3}{2}N_c < N_f < 3N_c$ maps over to the same interval $\frac{3}{2}\tilde{N}_c < N_f < 3\tilde{N}_c$. The extrema are exchanged and the fixed point is at $N_f = 2N_c = 2\tilde{N}_c$. Therefore both theories, in the IR are in a *Non-Abelian Coulomb Phase*. In the IR, the electric meson field M and the magnetic singlet T describe the same macroscopic field, since their dimensions match. Actually we are speaking about three different fixed points. At $\lambda = 0$ we have a fixed point at g^* where the singlet field T is free (it only has kinetic term). Therefore, at this point the dimension of the perturbation (1.188) is equal $1 + 2\frac{3}{2}\frac{N_c}{N_f} \leq 3$.

Hence it is a relevant perturbation, and it drives the theory in the IR to another fixed point (g^*, λ^*). It is this fixed point the one that describes the same physics as the electric theory. In particular, T and M flow to the same IR meson operator since both their quantum numbers and dimensions match. At the UV fixed point, however, there is not such a simple matching as with the barions. The UV dimensions of T and M disagree by an amount of one. In order to relate them we must introduce a scale μ

$$T^f{}_g = \frac{1}{\mu} M^f{}_g \equiv \frac{1}{\mu} Q^f \tilde{Q}_g \tag{1.191}$$

An independent way of introducing this independent scale is as the crossover scale of both asymptotically free theories whose dynamical scales are Λ and $\tilde{\Lambda}$.

$$\Lambda^{3N_c - N_f} \tilde{\Lambda}^{3\tilde{N}_c - N_f} = (-1)^{\tilde{N}_c} \mu^{N_f} \tag{1.192}$$

The behaviour of the RG flows embodied in this equation shows that the gauge coupling of one theory becomes weaker as the one of the dual becomes stronger.

Duality is an involution

The dual of $SU(\tilde{N}_c)$ and N_f flavours, contains again N_f flavours in the fundamental of the gauge group $SU(\tilde{\tilde{N}}_c) = SU(N_c)$. The relation (1.192) does not transform correctly. A second duality trasformation brings it to

$$\tilde{\Lambda}^{3\tilde{N}_c - N_f} \tilde{\tilde{\Lambda}}^{3\tilde{\tilde{N}}_c - N_f} = (-1)^{\tilde{\tilde{N}}_c} \tilde{\mu}^{N_f} \tag{1.193}$$

Since the final theory and the original gauge group are the same so are they scales and therefore (1.193) is the same as

$$\tilde{\Lambda}^{3\tilde{N}_c - N_f} \Lambda^{3N_c - N_f} = (-1)^{N_c} \tilde{\mu}^{N_f} \tag{1.194}$$

which, compared with (1.192) shows that $\tilde{\mu} = -\mu$.

Since the dual of the dual barions are the original ones, we expect the constituent quarks to be none other than the original $(Q^f, \tilde{Q}_f)$. In addition there is a gauge singlet $U^f{}_g$ that plays the role of the magnetic meson $M^{(m)}$, and shoud be identified with it (in the free UV) through a relation analogous to (1.191)

$$U^f{}_g = \frac{1}{\tilde{\mu}} \tilde{M}^f{}_g = \frac{1}{\tilde{\mu}} q^f \tilde{q}_g \tag{1.195}$$

The picture completes with the addition of the corresponding superpotential that adds up to the one written in (1.188), but expressed in the original variables through (1.191)

$$\begin{aligned} \mathcal{W} &= q_f T^f{}_g \tilde{q}^g + \tilde{Q}_f U^f{}_g Q^g \\ &= \frac{1}{\mu} \mathrm{tr} MT + \frac{1}{\tilde{\mu}} \tilde{Q}_f \tilde{M}^f{}_g Q^g \\ &= \frac{1}{\mu} \left(\mathrm{tr} M\tilde{M} - \tilde{Q}_f \tilde{M}^f{}_g Q^g \right) \end{aligned} \tag{1.196}$$

The first term is a mass term for both M and T which therefore may be integrated out through their equations fo motion.

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial M^f{}_g} &= \tilde{M}^g{}_f = 0 \\ \frac{\partial \mathcal{W}}{\partial \tilde{M}^f{}_g} &= M^g{}_f - Q^g \tilde{Q}_f = 0 \end{aligned} \tag{1.197}$$

Hence, with the advised equation $\tilde{\mu} = -\mu$ we obtain the desired relation $M = Q\tilde{Q}$ which makes the whole picture consistent.

Consistency with Deformation

v.e.v. deformation Turn on a *v.e.v.* $\langle M^1{}_1 \rangle = a^2$. For large a, the effect on the electric theory is tu Higgs the gauge group $SU(N_c) \hookrightarrow SU(N_c - 1)$.

mass deformation

1.6. Conifold Field Theory (Klebanov-Witten)

1.6.1. Free Theory

Let N_+ stand short for $N + M$. Consider an $N = 1$ QFT with $U(N_+) \times U(N)$ gauge symmetry, with (A_f, B_f) chiral superfields $f = 1, 2$. Hence this case falls in the general case analyzed before, where (A_1, B_1) and (A_2, B_2) build up two flavors $(Q_f, \tilde{Q}_f)$, $f = 1, 2 = N_f$. The gauge group is a direct product. Otherwise one should think of it as a unitary group with rank two Kronecker product matrices.

$$g \sim 1 + \alpha_A T^A \qquad T^A \left\{ \begin{array}{ll} T^A_{(N_+)} = t^A_{(N_+)} \otimes 1 & A = 1, ...N_+^2 \\ T^A_{(N)} = 1 \otimes t^A_{(N)} & A = 1, ..., N^2 \end{array} \right. \tag{1.198}$$

For $t^A_{(N_+)}$ and $t^A_{(N)}$ we may choose the standard basis

$$\begin{aligned} (t^{mn}_{(N_+)})^p{}_q &= \delta^{mp} \delta^n{}_q & m, n, p, q... = 1, ..., N_+ \\ (t^{ab}_{(N)})^c{}_d &= \delta^{ac} \delta^b{}_d & a, b, c, d, ... = 1, ..., N \end{aligned} \tag{1.199}$$

Otherwise indices $i, j, ...$ haver their rank restricted by the context. Correspondingly, A^{ma}_f, B^{nb}_g are rectangular matrices in color space. As representations of $U(N_+) \times U(N)$ the chiral superfields transform as

$$\left. \begin{array}{lll} A^{ma}_f & \to & (N_+, \bar{N}) \\ B^{nb}_g & \to & (\bar{N}_+, N) \end{array} \right\} \qquad m, n = 1, ..., N_+, a, b = 1, ..., N$$

$$\mathcal{L} = \frac{\tau}{16\pi i} \int d^2\theta \; W^2 + \frac{1}{4} \int d^2\theta d^2\bar{\theta} \left(A^{f\dagger} e^{2V} A_f + B^{g\dagger} e^{-V^t} B_g + \zeta V \right) \tag{1.200}$$

Vacuum solutions are given by the D flatness condition

$$\begin{aligned}\sum_A |D^A|^2 &= \sum_{A=1}^{N_+} \left| A^{f\dagger}_{ma}(t^A\otimes 1)^{ma}{}_{nb}A^{nb}_f + B^{f\dagger}_{ma}((-t^A)^t\otimes 1)^{ma}{}_{nb}B^{nb}_f\right|^2 \\ &+ \sum_{A=1}^{N} \left| A^{f\dagger}_{ma}(1\otimes(-t^A)^t)^{ma}{}_{nb}A^{nb}_f + B^{f\dagger}_{ma}\ (1\otimes t^A)^{ma}{}_{nb}\ B^{nb}_f\right|^2 \\ &= \sum_{m,n=1}^{N_+} \left| A^{f*}_{ma}A^{na}_f - B^{f*}_{ma}B^{na}_f\right|^2 + \sum_{a,b=1}^{N} \left| -A^{f*}_{ma}A^{mb}_f + B^{f*}_{ma}B^{mb}_f\right|^2\end{aligned}$$

where use of the standar basis for the Lie algebra $u(N)$, $(t^{mn})^i{}_j = \delta^{mi}\delta^n{}_j$ has been made. Performing a $SU(N_+)\times SU(N)$ rotation be may go to a diagonal basis

$$A^{ma}_f = a^{(a)}_f\delta^{ma} \qquad ; \qquad B^{nb}_f = b^{(b)}_f\delta^{nb} \tag{1.201}$$

where $(a^{(a)}_f, b^{(b)}_f)$ represent $4N$ complex parameters. We arrive at

$$\sum_{A=1} |D^A|^2 = 2\sum_a^N\sum_{f=1}^2 \left| |a^{(a)}_f|^2 - |b^{(a)}_f|^2\right|^2 = 0 \tag{1.202}$$

This imposes the vacuum conditions

$$\boxed{|a^{(r)}_1|^2 + |a^{(r)}_2|^2 - |b^{(r)}_1|^2 - |b^{(r)}_2|^2 = 0 \qquad r = 1,...,N} \tag{1.203}$$

- For generic diagonal *vevs.* like in (1.201) there is an unbroken $U(1)^N$ symmetry. The unbroken generators are the diagonally embedded Cartan subalgebra $t^{pp}_{(N_+)}\otimes 1 + 1\otimes t^{pp}_{(N)}$ with $(t^{pp})^i_j = \delta^{pi}\delta^p{}_j = \delta^i{}_j$.

$$\begin{aligned}&\Big[(t^{pp}\otimes 1)^{ma}{}_{nb}A^{nb}_f + ((-t^{pp})^t\otimes 1)^{ma}{}_{nb}B^{nb}_f \\ &+ \quad (1\otimes(-t^{pp})^t)^{ma}{}_{nb}A^{nb}_f + (1\otimes t^{pp})^{ma}{}_{nb}\ B^{nb}_f\Big] \\ &= \quad \Big[a^{(p)}_f - b^{(p)}_f - a^{(p)}_f + a^{(p)}_f\Big] \quad = \quad 0\end{aligned}$$

- When all eigenvalues are equal $a^{(r)}_f = a_f, b^{(r)}_f = b_f$ all diagonally embedded generators $[t^A_{(N_+)}\otimes 1 + 1\otimes t^A_{(N)}]$, $A = 1,...,N^2$ are unbroken, and the gauge symmetry is enhanced to $U(N)_V$.

1.6.2. Adding a Superpotential

Let us add the following $SU(2)\times SU(2)$ invariant tree level chiral interaction

$$\begin{aligned}\mathcal{W} &= \lambda\ \mathrm{tr}(A_dB_fA_gB_h)\epsilon^{dg}\epsilon^{fh} \\ &= 2\lambda\ \mathrm{tr}(A_1B_1A_2B_2 - A_1B_2A_2B_1)\end{aligned} \tag{1.204}$$

This represents a non-renormalizable interaction of mass dimension -4. Hence the coupling constant λ has mass dimension 1. The $\mathbf{F}-$term vacuum conditions read

$$\boxed{\begin{array}{lcl} \frac{\partial F}{\partial A_d^{ij}} = 0 & \rightarrow & B_1 A_g B_2 - B_2 A_g B_1 = 0 \\ \frac{\partial F}{\partial B_f^{kl}} = 0 & \rightarrow & A_1 B_h A_2 - A_2 B_h A_1 = 0 \end{array}} \tag{1.205}$$

1.6.3. Conifold Connection

The classical field theory is well aware that it represents branes moving on a conifold. Consider the diagonal form given in (1.201) for A_f and B^g. The F term equations (1.205) are trivially satisfied. Constraining $a_f^{(r)}$ and $b_g^{(r)}$ by means of equation (1.203) cuts from $8N$ down to $7N$ real variables. The diagonal action maximal abelian subalgebra of dimension N (so far we are having $U(N)$ instead of $SU(N)$) allows to mod out N phases. Hence $6N$ real or $3N$ complex variables remain. With them we can define $4N$ new complex numbers $w_A^{(r)}$ via

$$W_{fg}^{(r)} = a_f^{(r)} b_g^{(r)} = \frac{1}{\sqrt{2}} w_A^{(r)} \sigma_{fg}^A \qquad (A = 1, ..., 4)$$

constrained by the (in terms of a_f, b_g trivially satisfied) equation

$$\boxed{\det_{f,g} W_{fg}^{(r)} \;=\; 0 \;=\; \sum_{A=1}^{4} (w_A^{(r)})^2} \tag{1.206}$$

which is exactly the conifold equation. Therefore the $W_{fg}^{(r)}$ are coordinates that represent the position of the r'th $D3$-brane on the conifold.

1.6.4. Global Symmetries

With $\lambda = 0$ the model (1.200) enjoys the usual flavour $U(2) \times U(2) \sim SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A$ symmetry. the $SU(2)$ rotate the flavour indices A_f and B_g. As usual, $U(1)_A$ and $U(1)_R$ are anomalous. The two gauge groups have different beta functions $b = 3N_c - N_f$ (notice that (A_f, B_f) build up 2 (Dirac) flavours in total)

$$b^{SU(N_+)} = b = 3N_+ - 2N \quad ; \quad b^{SU(N)} = \tilde{b} = 3N - 2N_+$$

Correspondingly we have two different strong scales $\Lambda_{SU(N_+)} = \Lambda$ and $\Lambda_{SU(N)} = \tilde{\Lambda}$.
Consider the following table

	$SU(N_+)$	$SU(N)$	$SU(2)\times SU(2)$	$U(1)_B$	$U(1)_A$	$U(1)_R$
A_f	N_+	$\bar{N}$	$(2,1)$	$\frac{1}{2N_+N}$	$\frac{1}{2N_+N}$	$\frac{1}{2}$
B_f	$\bar{N}_+$	N	$(1,2)$	$\frac{-1}{2N_+N}$	$\frac{1}{2N_+N}$	$\frac{1}{2}$
Λ^b				0	$\frac{2}{N_+}$	$2M$
$\tilde{\Lambda}^{\tilde{b}}$				0	$\frac{2}{N}$	$-2M$
λ				0	$-\frac{2}{N_+N}$	0

The charges of $\Lambda, \bar{\Lambda}$ and λ are fixed by the conventional choice made for the charges of A_f and B_f. For example

- the $U(1)_R$ charges of have been selected to have a charge two superpotential with inert λ.
- The barionic and axial charges are conventional.
- The $U(1)_A$ and $U(1)_R$ charges of Λ and $\bar{\Lambda}$ are obtained as follows. Remember that the chiral anomaly of a single left-handed Weyl fermion $\psi_\alpha \to e^{i\alpha}\psi_\alpha$ shifts the θ parameter by

$$\theta \to \theta - T(R)\alpha \qquad T(R) = \begin{cases} 1 & R = N_c \text{ or } \bar{N}_c \\ 2N_c & R = \text{adjoint} \end{cases} \tag{1.207}$$

  We expect a total shift of the θ parameter given by $\theta \to \theta - n\alpha$ where n receives contributions from all the chiral fermions in the spectrum. To restore the symmetry one may endow θ with compensating transformations properties

$$\theta \to \theta + n\alpha \quad \Rightarrow \quad \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \quad \to \tau + \frac{n}{2\pi}\alpha \tag{1.208}$$

  The dynamical scale Λ also gets transformed

$$\Lambda^b = \mu e^{2\pi i\tau(\mu)} = \mu e^{-\frac{8\pi^2}{g^2}+i\theta} \quad \to \quad \Lambda^b e^{in\alpha} \tag{1.209}$$

  From the point of view of $SU(N_+)$, the fields A_f^{ma}, B_f^{ma} contain $4N$ (indices $f = 1,2; a = 1,...,N$)) weyl quarks q_A, q_B with charge $\frac{1}{2N_+N}$ in the fundamental (index $m = 1,...,N_+$). This yields the $U(1)_A$ charge of Λ^b as $\frac{2}{N_+}$. The $U(1)_R$ charge of chiral quarks q_A, q_B is $\frac{1}{2} - 1 = -\frac{1}{2}$. Again there are $4N$ of them. But now we must count the gauginos λ^α with charge 1 in the adjoint of $SU(N_+)$, and they contribute $2N_+$. Altogether $2N_+ - \frac{1}{2}\,4N = 2M$. The analysis for $SU(N)$ is the same.

From the list of charges above we notice that $U(1)_R$ transformations with phase

$$e^{2\pi i \frac{m}{2M}} \in \mathbf{Z_{2M}} \quad ; m = 1,2,....,2M$$

affect $(A_f, B_g) \to e^{\pi i \frac{m}{2M}}(A_f, B_g)$ whereas Λ^b and $\tilde{\Lambda}^{\tilde{b}}$ change by $e^{\pm 2\pi i \frac{m}{2M}2M} = 1$, *ie* $\Lambda, \tilde{\Lambda}$ and λ are unchanged. This is the anomaly free $\mathbf{Z}_{2M}$ remant of $U(1)_R$. For $M = 0$ the full

$U(1)_R$ is anomaly free. Ultimately this justifies the choice of charges 1/2 for the chiral multiplets.

Apart from the superpotential

$$\mathcal{W} = \lambda \ \mathrm{tr}(A_d B_f A_g B_h)\epsilon^{dg}\epsilon^{fh}$$

there are other combinations of parameters and fields which are invariant under the global symmetries. For example

$$\begin{aligned} I &= \lambda^{3M} \frac{\tilde{\Lambda}^{\tilde{b}}}{\Lambda^b} \left[\mathrm{tr}(A_d B_f A_g B_h)\epsilon^{dg}\epsilon^{fh}\right]^{2M} \\ &= \lambda^{M} \frac{\tilde{\Lambda}^{\tilde{b}}}{\Lambda^b} W^{2M} \end{aligned} \tag{1.210}$$

Another possibility comes from quotients of the form

$$R^{(1)} = \frac{\mathrm{tr}(A_d B_f)\mathrm{tr}(A_g B_h)\epsilon^{dg}\epsilon^{fh}}{\mathrm{tr}(A_d B_f A_g B_h)\epsilon^{dg}\epsilon^{fh}} \tag{1.211}$$

with equal number of fields A and B in numerator and denominator but differently contracted índices. There i salso an important invariant involving just scalar parameters

$$J = \lambda^{N_+ + N} \Lambda^b \tilde{\Lambda}^{\tilde{b}}. \tag{1.212}$$

Its logarithm plays the analog of the dimensionless coupling constant τ in $\mathcal{N} = 4$ SYM.

As a general rule, the tree level superpotential will be renormalized to take the following form

$$\boxed{\mathcal{W}_{eff} \ \ \hookrightarrow \ \ \lambda \ \ \mathrm{tr}(A_d B_f A_g B_h)\epsilon^{dg}\epsilon^{fh} \ F(I, J, R^{(s)})} \tag{1.213}$$

for a, to be determined function F.

Capítulo 2

Selected Topics in Gauge Theories

2.1. Anomalies in Gauge Theories

Classical symmetries are symmetries of the Lagrangian. Quantum symmetries are symmetries of the path integral. Whenever a classical symmetry is not a quantum symmetry it is termed anomalous. The prototypical example is the path integral of a chiarl Weyl fermion ψ coupled to a gauge field A^a.

$$S[\psi, A_\mu] = \int d^4x\, i\, \psi^\dagger \bar{\sigma}^\mu (\partial_\mu + i A^a_\mu T^a)\psi \tag{2.1}$$

Apart from the fact that there is a non-abelian gauge symmetry that involves transforming A^a_μ, there also exist a global chiral $U(1)$ symmetry whereby $\psi \to e^{i\alpha}\psi$ gets rotated by a global phase. An clever way to capture the Noether current is to make a *local* transformation, $\alpha \to \alpha(x)$, instead of a global one. This is not a symmetry, hence he action changes by

$$S[e^{i\alpha(x)}\psi, A_\mu] = S[\psi, A_\mu] + i \int d^4x (i\alpha(x))\partial_\mu(\psi^\dagger\bar{\sigma}^\mu\psi)\,. \tag{2.2}$$

Not for constant α this must be a symmetry, hence the additional term must vanish, or $\partial_\mu j^\mu_A = 0$ with

$$j^\mu_A = i\psi^\dagger\bar{\sigma}^\mu\psi \tag{2.3}$$

The Fujikawa method allows to compute the violation of the conservation equation, so that quantum mechanically it is replaced by

$$\partial_\mu j^\mu_A = \frac{1}{32\pi^2} F^{a\mu\nu}F^a_{\mu\nu} \tag{2.4}$$

The easiest way is to regard the path integral as a function of the background gauge field

$$Z[A_\mu] = \int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{-S[\psi,A_\mu]} = \int \mathcal{D}(e^{i\alpha}\psi)\mathcal{D}(e^{-i\alpha}\psi^\dagger)e^{-S[e^{i\alpha}\psi,A_\mu]} \tag{2.5}$$

Again, letting $\alpha(x)$ be local, we expect that

$$\int \mathcal{D}(e^{i\alpha(x)}\psi)\mathcal{D}(e^{-i\alpha(x)}\psi^\dagger)e^{-S[e^{i\alpha}(x)\psi,A_\mu]} \;=\; \int \mathcal{D}\psi\mathcal{D}\psi^\dagger J[\alpha(x)]e^{-S[\psi,A_\mu]-\int d^4x i\alpha(x)\partial_\mu j^\mu_A}$$

$$= \int \mathcal{D}\psi \mathcal{D}\psi^\dagger e^{-S[\psi, A_\mu] - \int d^4x i\alpha(x)(\partial_\mu j^\mu_A - A(x))}$$

where $A(x) = \exp[\log J(x)]$ and $J(x)$ is a Jacobian determinant associated to the change of measure. Since Z is independent of $\alpha(x)$ we may take functional derivative to obtain the new conservation law

$$\left.\frac{\delta Z}{\delta i\alpha(x)}\right|_{\alpha=0} = \langle \partial_\mu j^\mu(x) - A(x) \rangle = 0 \tag{2.6}$$

where the brackets stand for expectation values. A carefull evaluation of $A(x)$ was first done by Fujikawa, giving the following result

$$A(x) = \frac{1}{16\pi^2} \mathrm{Tr} F^{\mu\nu} \tilde{F}_{\mu\nu} \tag{2.7}$$

Capítulo 3

The Minimal Supersymmetric Standard Model

3.1. Introduction

If supersymmetry was a real symmetry of the world, the spectrum of particles should come in representations of the algebra. In these representations, fermions and bosons are grouped together with equal masses. In 1976-77 Pierre Fayet made a first attempt in this direction. In a sense this program resembles the one followed by P.A.M. Dirac when postulating that the proton would be the anti-particle of the electron. So, for example the photon γ and the neutrino ν, seem to be suited to be inside the same vector multiplet. Other possible susy pairs would be such as $(W^{\pm}, e^{\pm})$.

After realising that the program was failure, the standard point of view now is to accept that the susy partners have never been observed, and therefore, SUSY, if there, must be broken at the energy scale of 1 TeV.

3.1.0.1 The hierarchy problem

In high energy physics there are at least two fundamental scales: the Planck mass $M_{Pl} \sim 10^{19}$ GeV defining the scale of quantum gravity, and the electroweak scale $M_{ew} \sim 10^{2}$ GeV, defining the electroweak symmetry breaking scale. The obvious questions to solve are

- Why $M_{ew} << M_{Pl}$? This is the hiearchy problem
- Even if this hierarchy was postulated classically, is it stable? This is the "naturalness"problem

To understand the naturalness problem, let us review the fate of masses in the Standard model.

Gauge particles are massless because of gauge invariance. A term such as $m_A A_\mu A^\mu$ is not invariant under $A_\mu \to A_\mu + \partial_\mu \alpha$.

Fermions are also massless in the Standard model because of of chirality. This means that $m\bar{\psi}_L\psi_R$ is not gauge invariant because tipically ψ_L and ψ_R live in different (conjugate) representations of the gauge group. The receive masses through the Higgs mechanism (e.g. $H \cdot \psi\psi$ gives mass to ψ after H gets a v.e.v.).

The Higgs, as any scalar particle can have a mass term $m_H^2 \bar{H} H$ which is not forbidden by any symmetry. Moreover, loop corrections are quadratic, hence they will shift the value of m_H up to the cutoff scale Λ of the theory. This would ruin the classical hierarcy $m_H \sim M_{ew} << M_{Pl}$.

Supersymmetry solves the naturalness problem by adding partners that, when running in the loops, cancel the quadratic divergences. This is the non-renormalization theorem at work. Hence supersymmetry does not explain the hierarchy, but once given, stabilizes it.

3.2. The MSSM

Matter in the standard model is chiral. This means that L and R chiralities transform under different representations of the gauge group. It is customary to list only left handed chiralities.

	SU(3)×SU(2)×U(1)	ψ
$Q = \begin{pmatrix} U \\ D \end{pmatrix}$	(3 , 2 , 1/3)	$(q_L, \tilde{q}_L, F_{q_R})$
$L = \begin{pmatrix} N \\ E \end{pmatrix}$	(1 , 2 , -1)	$(l_L, \tilde{l}_R, F_{l_R})$
U^c	($\bar{3}$, 1 , -4/3)	$(u_L^c, \tilde{u}_R^*, F_{u_R}^*)$
D^c	($\bar{3}$, 1 , 2/3)	$(d_L^c, \tilde{d}_R^*, F_{d_R}^*)$
E^c	(1 , 1 , 2)	$(e_L^c, \tilde{e}_R^*, F_{e_R}^*)$

3.2.1. Field content

The vector multiplets include now new fermions: gauginos and higgsinos.

$$W^{\pm} = (A_W^{\pm}, \lambda_W^{\pm}, D^{\pm}) \quad (3.1)$$

$$W^0 = (A_W^0, \lambda_W^0, D^0) \quad (3.2)$$

$$A = (A, \lambda, D) \quad (3.3)$$

This puts in danger the delicate cancellation of gauge anomalies that occurs in the Standard Model.

- Gauginos pose no problem as they couple vectorially
- a single Higgsino running in a triangle loop contributes proportionally to $Y_H^3 = +1^3$ to the gauge anomaly.

A solution is to introduce a second Higgs doublet of opposite hypercharge such that $Y_{H_1}^3 + Y_{H_2}^3 = (+1)^3 + (-1)^3 = 0$.

Also this is required by the supersymmetric version of the Higgs mechanism. In this mechanism, the starting point is a massless vector supermultiplet and a chiral massless supermultiplet

Massless vector	Massless chiral
$V \to (A_{\perp 1,2} \,;\, \lambda_{1,2})$	$H \to (h_0 + ih_1 \,,\, \psi_{h_{1,2}})$

After the Higgs mechanism, one of the scalars, say h_0 provide the third longitudinal d.o.f for the vector potential. A massive supermultiplet has, in addition, four spin 1/2 d.o.f and an additional scalar

Massive vector
$V \to (A_{\perp 1,2} \,,\, A_{\parallel} \sim h_0 \,;\, \lambda_{1,2} \,,\, \chi_{1,2} \sim \psi_{h_{1,2}} \,,\, \phi \sim h_1)$

In summary, for each vector supermultiplet we need 2 bosonic degrees of freedom. Then $(W^{\pm}, Z^0)$ needs 6 bosonic degrees of freedoms. In a $SU(2)$ doublet H_i, $i = 1, 2$ there are only 4. Let us introduce therefore a second doublet

	SU(3)× SU(2)×U(1)
$H_1 = \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix}$	(1 , 2 , 1)
$H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}$	(1 , 2 , -1)

such that, after giving mass to all vector bosons, still 2 degrees of freedom are left over. One is a scalar h^0, and the other one a pseudoescalar A^0.
This is the MSSM content.

3.2.2. Couplings

The superpotential contains the relevant couplings. On one hand we need the Higgs to break $SU(2) \times U(1)_Y \to U(1)_{em}$. We can write bilinear couplings

$$W_H = -\mu H_1 \cdot H_2 = -\mu \epsilon_{ij} H_i H_j \tag{3.4}$$

μ has mass dimension 1, hence its smallness, as compared with other scales like unificaton scale or Planck scale should be given a good reason (μ problem).
Cubic terms are needed to give Yukawa masses to the particles of the Standard Model

$$W_{Yuk} = \lambda_d Q \cdot H_1 D^c + \lambda_u Q \cdot H_2 U^c + \lambda_e L \cdot H_1 E^c \tag{3.5}$$

with color indices suppressed.
We shall assume that the only scalars that acquire v.e.v. are the (electrically) neutral Higgses $H_{1,2}^0$, i.e.

$$\langle H_1 \rangle = \begin{pmatrix} \langle H_1^0 \rangle \\ 0 \end{pmatrix} \quad ; \quad \langle H_2 \rangle = \begin{pmatrix} 0 \\ \langle H_1^0 \rangle \end{pmatrix} \tag{3.6}$$

then

$$W_{Yuk} = -\lambda_d \langle H_1^0 \rangle D_L D_L^c + \lambda_u \langle H_2^0 \rangle U_L U_L^c - \lambda_e \langle H_1^0 \rangle E_L E_L^c \tag{3.7}$$

From here, the lagrangian in components has terms

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\sum_{ij}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\bar{\psi}^c_{i,L}\psi_{j,L} + h.c. + ... \\ &= \lambda_d\langle H_1^0\rangle\bar{\psi}^c_{d_L}\psi_{d_L} + \lambda_u\langle H_2^0\rangle\bar{\psi}^c_{u_L}\psi_{u_L} + \lambda_e\langle H_1^0\rangle\bar{\psi}^c_{e_L}\psi_{e_L} + ...\end{aligned} \tag{3.8}$$

Notice that hypercharge conservation demands H_2 to give mass to up quarks ψ_u, since we cannot make use of H_1^*. This is another reason to introduce two Higgs doublets.

3.2.3. R-parity

3.2.3.1 Dangerous couplings

In the Standard Model, gauge invariance plus renormalizability restricts the Yukawa couplings to be ones in the previous section. In the MSSM this is not so, as there are more particles involved. For example, the following is a set of possible gauge invariant renormalizable couplings, $L \cdot LE^c$, $Q \cdot LD^c$ and $U^cD^cD^c$. They are dangerous because the violate lepton or baryon number.

	$L \cdot LE^c$	$Q \cdot LD^c$	$U^cD^cD^c$
$U_1(B)$	0 + 0 + 0 = 0	1 + 0 - 1 = 0	-1-1-1 = - 3
$U_1(L)$	1 + 1 - 1 = 1	0 + 1 + 0 = 1	0 + 0 + 0 = 0

An important consequence would be the decay of the proton, through the Feynman diagram shown in fig. 3.1

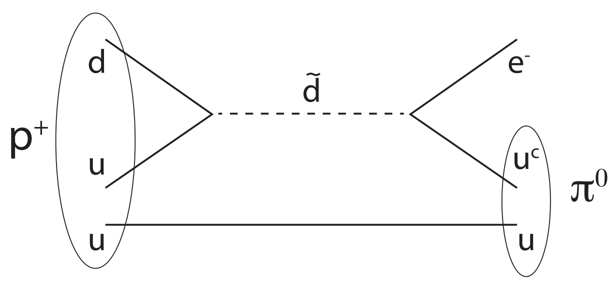

Figura 3.1: Proton decay mediate through the terms $U^cD^cD^c$ and $Q \cdot LD^c$ in the superpotential.

Since we now experimentally that the proton lifetime is very long $\tau > 10^{25}$ years, this poses a bound on the mass of the intermediary particle $m_{\tilde{d}} \geq 10^{15}$GeV. For trilinear couplings of $\mathcal{O}(1)$ this implies that supersymmetry is so badly broken it will be useless as a possible solution to the hierarchy problem.

We shall assume they are absent in the MSSM. But a more refined way to rule them out is to postulate a new symmetry that forbids them. It must be a symmetry that treats differently the SM particles from theis SUSY partners. Hence it better be a subgroup of the R-symmetry called R-parity. Let r be such that

$$\pi_R : \theta \to e^{i\pi}\theta = -\,\theta \qquad \Rightarrow \qquad R(\theta) = -1 \tag{3.9}$$

we say that θ is parity odd. Let us assign definite parities to the matter multiplets

$$\begin{aligned} &R(Q) = R(L) = R(U) = R(D) = R(E) = -1 \\ &R(H_1) = R(H_2) = 1 \end{aligned} \tag{3.10}$$

then

$$R(\lambda_d Q \cdot H_1 D^c) = R(\lambda_u Q \cdot H_2 U^c) = R(\lambda_e L \cdot H_1 E^c) = +1 \tag{3.11}$$

are allowed whereas

$$R(L \cdot L E^c) = R(Q \cdot L D^c) = R(U^c D^c D^c) = -1 \tag{3.12}$$

3.2.3.2 Experimental consequences

There is a compact way to assign R parity to component fields. Given its baryon number is B, lepton number is L and spin is s,

$$R = e^{i\pi(3B+L+2s)} \tag{3.13}$$

This implies that all susy partners have odd parity

	B	L	S	3B+2L+S	R
q	1/3	0	1/2	2	1
$\tilde{q}$	1/3	0	0	1	-1
l	0	1	1/2	2	1
$\tilde{l}$	0	1	0	1	-1

Concerning experimental consequences we observe that:

- Supersymmetric particles have to come in pairs, since $+1 = (-1)(-1)$.
- Among al the supersymmetric partners, there must be one that is the lightest (LSP). Having R = -1 it cannot decay anymore and therefore it must be stable. If the LSP is neutral (neutralino, higgsino, photino), then it is a weakly interacting massive particle (WIMP), and is a candidate for dark matter.

3.3. Electroweak symmetry breaking

In order to discuss gauge symmetry breaking we are entitled to analyze the full scalar potential. Now, in contrast to the SM, lots of scalar fields could acquire a v.e.v.. In order not to break color or R-symmetry we will set them all to vanishing v.e.v. except for the Higgs fields. Hence, the scalar potential is

$$V(H_1^i, H_2^i) = V_F + V_D \tag{3.14}$$

For example

$$V_F = \sum_i \left(|F_1^i|^2 + |F_2^i|^2\right) = \sum_i \left(\left|\frac{\partial W}{\partial H_1^i}\right|^2 + \left|\frac{\partial W}{\partial H_2^i}\right|^2 \right)$$
$$= \sum_{ij} |\mu|^2 \epsilon_{ij} H_2^{j*} \epsilon_{ik} H_2^k + (1 \leftrightarrow 2)$$
$$= |\mu|^2 \left(H_1^\dagger H_1 + H_2^\dagger H_2 \right) \tag{3.15}$$

The D term potential involves contributions from both the $SU(2)$ and the $U(1)_Y$ gauge vector supermultiplets

$$V_D = \frac{1}{2} g^2 \left(H_1^\dagger \frac{\vec{\tau}}{2} H_1 + H_2^\dagger \frac{\vec{\tau}}{2} H_2 \right)^2 + \frac{1}{2} \left(\frac{g'}{2}\right)^2 \left(-H_1^\dagger H_1 + H_2^\dagger H_2 \right)^2 . \tag{3.16}$$

where g and $g'/2$ are, respectively, the couplings of $SU(2)$ and $U(1)_Y$.

The quadratic contribution (3.15) is positive definite, and the quartic couplings in (3.16) are of order $\sim g^2$ hence small. Hence, if at all there is SSB, it would happen at a very small scale $M_W^2 \sim g^2 v^2$.

The crucial assumption is that spontaneous SUSY breaking induces new terms that trigger gauge SSB at a desired pattern. These include masses for scalar particles, as well as bilinear couplings

$$V_{SB} = m_{H_1}^2 H_1^\dagger H_1 + m_{H_2}^2 H_2^\dagger H_2 + (B_\mu H_1 \cdot H_2 + h.c.) . \tag{3.17}$$

One can show that the minimization of $V_F + V_D + V_{SB}$ happens for $\langle H_{1,2}^\pm \rangle = 0$ which implies $U(1)_{em}$ will not be broken. Therefore we can restrict ourselves to the neutral higgs components.

$$V_{SB}(H_1^0, H_2^0) = m_1^2 {H_1}^2 |H_1^0|^2 + m_2^2 {H_2}^2 |H_2^0|^2 + B_\mu (H_1^0 H_2^0 + H_1^{0*} H_2^{0*})$$
$$+ \frac{g^2 + g'^2}{8} (|H_1^0|^2 - |H_2^0|^2)^2 \tag{3.18}$$

were we have introduced the mass parameter

$$m_1^2 = m_{H_1}^2 + |\mu|^2$$
$$m_2^2 = m_{H_2}^2 + |\mu|^2 \tag{3.19}$$

These parameters must be constrained by some physical requirements.

- Stability

$$m_1^2 + m_2^2 > 2|B_\mu|$$

- Gauge symmetry breaking $\Rightarrow H_1^0 = H_2^0 = 0$ should not be a minimum

$$m_1^2 m_2^2 < |B_\mu|^2 .$$

These two conditions imply that necessarily $m_1^2 \neq m_2^2$.
Defining

$$v_1 = \langle H_1^0 \rangle \quad ; \quad v_2 = \langle H_2^0 \rangle$$

the relevant magnitudes are

$$v^2 = v_1^2 + v_2^2 \quad ; \quad \tan\beta = \frac{v_2}{v_1}$$

in terms of which, the physical masses of the gauge bosons can be expressed as

$$M_W^2 = \frac{g^2}{2}(v_1^2 + v_2^2) \quad ; \quad M_Z^2 = 2\frac{m_1^2 - m_2^2 \tan^2\beta}{\tan^2\beta - 1} \tag{3.20}$$

3.4. Supersymmetry Breaking

If SUSY is true, it must be broken. The mass of the lightest squark must be well above that of the heaviest quark in the SM. However if we supersymmetry is spontaneously borken inside the MSSM, we have the condition

$$\mathrm{Str} M^2 = \mathrm{Tr}(-1)^F M^2 = \mathrm{Tr} M^2_{scalar} - \mathrm{Tr} M^2_{fermions} = 0\,. \tag{3.21}$$

Hence, in mean, scalars are as light as fermions. The way out of this concundrum, is to postulate the existence of a hidden sector where the SUSY SSB occurs. The particles in this sector couple very weakly to those of the SM. Hence on one side they are difficult to rule out and, on the other, the effects of the SUSY breaking is weakly mediated to the observable sector.

There are three basic scenarios for this mediation

- *Gravitational mediation.*

  Gravitons couple both sectors, and therefore, loop effects are suppressed by $1/M_{Planck} \sim 10^{-18}$GeV. In order to obtain a mass splitting, the amount of susy breaking we need, by mere dimensional analysis will be

  $$\Delta m = \frac{M^2_{\not{susy}}}{M_{Planck}} \sim 1\,\mathrm{TeV} = 10^3 GeV \tag{3.22}$$

  hence $M_{\not{susy}} \sim \sqrt{\Delta m\, M_{Planck}} \sim 10^{10}$ GeV.

- *Gauge mediation*

  $$G = (SU(3) \times SU(2) \times U(1)) \times G_{\not{susy}} \equiv G_0 \times G_{\not{susy}}$$

  Matter fields are charged under both G_0 and $G_{\not{susy}}$ which gives a $M_{\not{susy}}$ of order Δm i.e.E $\mathcal{O}(1)$ TeV. In that case, the gravitino mass $m_{3/2}$ is given by

  $$m_{3/2} \sim \frac{M^2_{\not{susy}}}{M_{Planck}} \sim 10^{-3} eV\,.$$

- *Anomaly mediation*

  This is mediated by some auxiliary fields of supergravity. Effects are suppressed by loop counting.

3.4.0.3 Soft Susy Breaking terms

In all cases, the low energy effective lagrangian for the observable sector develops what is so called: soft supersymmetry breaking terms

$$L_{\not{susy},soft} = m_\phi^2 \phi^* \phi + (M_\lambda \lambda\lambda + h.c.) + (A\phi^3 + h.c). \tag{3.23}$$

i.e. masses and trilinear couplings for the scalars, and mass terms for the gauginos.

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