Lecture 10: Assorted string dualities

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STRING THEORY

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Cremmer-Julia-Scherk supergravity in 11D

The bosonic part of the action is simple since there are only two fields present:

- the metric, G_{MN} and
- a 4-form field strength, $G_{[4]} = dD_{[3]}$.

We use conventions for the *n*-forms such that

$$G_{[n]} := \frac{1}{n!} G_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} .$$

The action reads:

$$\mathcal{S}_{\mathrm{CJS}} = \frac{1}{16\pi G_{11}^N} \left(\int d^{11}x \sqrt{-G} \left[R - \frac{1}{48} {G_{[4]}}^2 \right] + \frac{1}{6} \int D_{[3]} \wedge G_{[4]} \wedge G_{[4]} \right) .$$

G_{11}^N defines the 11D Planck length by $G_{11}^N = \ell_p^9$.

The last term is the Chern-Simons-like term necessary for supersymmetry to hold. This Lagrangian does not admit a cosmological term.

Dimensional reduction on a circle: field content

Let us start with a theory in $D+1$ dimensions. We can recast the metric as

$$ds^2 = G_{MN}\, dz^M\, dz^N = g_{\mu\nu}\, dx^\mu\, dx^\nu + e^{\frac{4}{3}\Phi}\,(dy + A_\mu\, dx^\mu)(dy + A_\nu\, dx^\nu)\ ,$$

where $z^M = (x^\mu, y)$, $\mu = 0 \ldots D-1$. Assume that all functions depend only on x^μ (zero modes in the internal direction).

The components of the metric G_{MN} read:

$$G_{\mu\nu} = g_{\mu\nu} + e^{\frac{4}{3}\Phi}\, A_\mu\, A_\nu\ , \qquad G_{\mu D} = e^{\frac{4}{3}\Phi}\, A_\mu\ , \qquad G_{DD} = e^{\frac{4}{3}\Phi} = \frac{R_{11}^2}{\ell_P^2}\ .$$

The inverse metric components G^{MN}, such that $G^{MP}\, G_{PN} = \delta^M{}_N$, are:

$$G^{\mu\nu} = g^{\mu\nu}\ , \qquad G^{\mu D} = -A^\mu := -g^{\mu\nu}\, A_\nu\ , \qquad G^{DD} = e^{-\frac{4}{3}\Phi} + A^\lambda\, A_\lambda\ .$$

where $g^{\mu\nu}$ is such that $g^{\mu\lambda}\, g_{\lambda\nu} = \delta^\mu{}_\nu$. The relation between the determinants is straightforward, namely, $G = g\, e^{\frac{4}{3}\Phi}$.

Upon dimensional reduction along a circle, $G_{MN} \to g_{\mu\nu}$, Φ, A_μ.

Dimensional reduction on a circle: field content

We have to see how the form $G_{[4]}$ or, instead, its potential $D_{[3]}$, reduces upon compactification on a circle: it gives rise to two potentials $C_{[3]}$ and $C_{[2]}$:

$$D_{[3]} = C_{[3]} + C_{[2]} \wedge dy \ .$$

The corresponding field strengths are easily found

$$G_{[4]} = dD_{[3]} = dC_{[3]} + dC_{[2]} \wedge dy = F_{[4]} + F_{[3]} \wedge dy \ .$$

The Lagrangian density $G_{[4]} \wedge {}^\star G_{[4]}$ can be written as

$$\frac{1}{4!} G_{[4]} \wedge {}^\star G_{[4]} = \frac{1}{4!} \widetilde{F}_{[4]} \wedge {}^\star \widetilde{F}_{[4]} + \frac{1}{3!} e^{-2\Phi} F_{[3]} \wedge {}^\star F_{[3]} \ ,$$

where $\widetilde{F}_{[4]} = F_{[4]} + A_{[1]} \wedge F_{[3]}$, and $A_{[1]} := A_\mu \, dx^\mu$.

The appearance of Φ and A_μ should not be surprising: they come from *.

Upon dimensional reduction along a circle, $D_{[3]} \to C_{[2]}, C_{[3]}$.

Dimensional reduction: dynamics

We shall see how the dynamics looks like upon dimensional reduction,

$$\mathcal{S} = \frac{1}{16\pi G^N_{D+1}} \int d^{D+1}z \, \sqrt{-G} \, R[G] \, ,$$

where $R[G]$ is the curvature scalar for G_{MN}.

The curvature scalar can also be written in terms of D-dimensional quantities,

$$R[G] = R[g] + 2\,\partial_\mu \Phi \, \partial^\mu \Phi - e^{-2\Phi} \Box \, e^{2\Phi} - \frac{1}{4} \, e^{2\Phi} \, F_{[2]} \wedge {}^\star F_{[2]} \, ,$$

where $R[g]$ and $\Box$ are built from the metric $g_{\mu\nu}$ and $F_{[2]} = dA_{[1]}$.

The action then reads, when we set $D = 10$,

$$\mathcal{S} = \frac{1}{16\pi G^N_{10}} \left[\int d^{10}x \, \sqrt{-g} \, e^{-2\Phi} \left(R + 4\,\partial_\mu \Phi \, \partial^\mu \Phi \right) - \frac{1}{4} \int F_{[2]} \wedge {}^\star F_{[2]} \right] ,$$

where we have defined $G^N_{10} \sim G^N_{11} / R_{11}$.

We shall now focus on the second term in the Cremmer-Julia-Scherk action,

$$\mathcal{S} = \frac{1}{16\pi G_{11}^N}\left(-\frac{1}{2}\right)\frac{1}{4!}\int G_{[4]} \wedge {}^\star G_{[4]} \ .$$

But we have already obtained the integrand in ten dimensional language:

$$\mathcal{S} = \frac{1}{16\pi G_{10}^N}\left(-\frac{1}{2}\right)\left[\frac{1}{4!}\int \widetilde{F}_{[4]} \wedge {}^\star \widetilde{F}_{[4]} + \frac{1}{3!}\int e^{-2\Phi} H_{[3]} \wedge {}^\star H_{[3]}\right] ;$$

here we have conveniently renamed the 3-form, $F_{[3]} \ \rightarrow \ H_{[3]}$.

Finally, the Chern-Simons term:

$$\mathcal{S}_{\mathrm{CS}} = \frac{1}{16\pi G_{11}^N}\,\frac{1}{6}\int D_{[3]} \wedge G_{[4]} \wedge G_{[4]} \ ,$$

since $G_{[4]} = F_{[4]} + H_{[3]} \wedge dy$ and $D_{[3]} = C_{[3]} + B_{[2]} \wedge dy$, gives:

$$\mathcal{S}_{\mathrm{CS}} = \frac{1}{16\pi G_{10}^N}\,\frac{1}{2}\int B_{[2]} \wedge F_{[4]} \wedge F_{[4]} \ ,$$

where, again, $C_{[2]} \ \rightarrow \ B_{[2]}$.

Adding the three pieces altogether we get

$$\mathcal{S} = \frac{1}{16\pi G_{10}^N} \int d^{10}x \sqrt{-g}\, e^{-2\Phi} \left(R + 4\, \partial_\mu \Phi\, \partial^\mu \Phi - \frac{1}{12} {H_{[3]}}^2 \right)$$

$$- \frac{1}{32\pi G_{10}^N} \int \left[\frac{1}{2!} F_{[2]} \wedge {}^\star F_{[2]} + \frac{1}{4!} \widetilde{F}_{[4]} \wedge {}^\star \widetilde{F}_{[4]} - B_{[2]} \wedge F_{[4]} \wedge F_{[4]} \right] ,$$

which is nothing but the type IIA supergravity action!

Notice that the dilaton field Φ is related to the $11^{\rm th}$ dimensional radius, R_{11},

$$e^{\frac{2}{3}\Phi_0} \sim \frac{R_{11}}{\ell_P} \qquad \Longrightarrow \qquad R_{11} \sim e^{\frac{2}{3}\Phi_0}\, \ell_P \; .$$

The string units are given by the string length, $\ell_s = \sqrt{\alpha'}$. Now, we have

$$e^{2\Phi_0}\, {\ell_s}^8 \sim G_{10}^N \sim \frac{G_{11}^N}{R_{11}} = \frac{{\ell_P}^9}{R_{11}} \quad \Rightarrow \quad \ell_P \sim e^{\frac{1}{3}\Phi_0}\, \ell_s \quad \Rightarrow \quad R_{11} \sim e^{\Phi_0}\, \ell_s \; .$$

M-theory

We have seen, then, that

$$R_{11} \sim e^{\Phi_0} \ell_s \ .$$

As we will see in the coming lectures, when discussing perturbative string amplitudes, the expectation value of the dilaton provides the string coupling,

$$g_s = e^{\Phi_0} \ .$$

Thus, we see that at weak string coupling,

$$g_s \ll 1 \qquad \Rightarrow \qquad R_{11} \ll \ell_s \ .$$

As well, $\ell_P \sim g_s^{1/3} \ell_s$, thus $\ell_P \ll \ell_s$. These quantities are also small compared to the 10D Planck scale, $L_p = G_{10}^{N\ 1/8} \sim g_s^{1/4} \ell_s$.

Perturbative string theory is thus consistently described in ten dimensions.

However, the $g_s \gg 1$ limit of these expressions points towards the existence of an eleven dimensional strongly coupled regime!

M-theory branes

What is the Lagrangian of this 11D theory? We don't know much about it!

However, we know that its low energy limit must be eleven dimensional supergravity, whose (bosonic) field content is very simple.

It has a single $G_{[4]}$; two objects can couple to its potential:

- an electric $M2$-brane, or,
- a magnetic $M5$-brane.

Eleven dimensional supergravity is an $\mathcal{N} = 1$ theory with 32 supercharges,

$$\{\bar{Q}_\alpha, Q_\beta\} = (\Gamma^M C)_{\alpha\beta}\, P_M + (\Gamma^{MN} C)_{\alpha\beta}\, Z_{MN} + (\Gamma^{MNPQR} C)_{\alpha\beta}\, Z_{MNPQR} \,.$$

We have seen earlier this year that SUSY algebras have special multiplets called BPS. Their mass is bound to be equal to the (absolute value of the) eigenvalues of the central extensions. Here, they have Lorentz indices:

The BPS states must be extended objects: 2- and 5-dimensional!

M-theory branes

There must be solutions of 11D supergravity preserving one half of the supersymmetries that correspond to the *M*2-brane and to the *M*5-brane.

Since there is no dilaton, they are easier to find than the *p*-branes we found earlier. The two solutions are thus:

$$ds^2_{M2} = \left(1 + \frac{1}{6}\frac{Q_2}{r^6}\right)^{-\frac{2}{3}} \left(-dt^2 + d\mathbf{y}^2_{(2)}\right) + \left(1 + \frac{1}{6}\frac{Q_2}{r^6}\right)^{\frac{1}{3}} d\mathbf{x}^2_{(8)} \ ,$$

with $F_{ty_1y_2r} = (H^{-1})'$, and:

$$ds^2_{M5} = \left(1 + \frac{1}{3}\frac{Q_5}{r^3}\right)^{-\frac{1}{3}} \left(-dt^2 + d\mathbf{y}^2_{(5)}\right) + \left(1 + \frac{1}{3}\frac{Q_5}{r^3}\right)^{\frac{2}{3}} d\mathbf{x}^2_{(5)} \ ,$$

with $F_{\theta_1\ldots\theta_4} = Q_5\ \omega_4$.

Interestingly enough, both solutions have a suggestive *near-horizon limit*:

The *M*2-brane $\to$ $AdS_4 \times S^7$, while the *M*5-brane $\to$ $AdS_7 \times S^4$.

M-theory branes and D-branes of type IIA

The tension of the M-branes can be computed. They have to be proportional to specific powers of the inverse of ℓ_P; indeed,

$$T_{M2} = \frac{\pi^{1/3}}{2^{2/3}\,{\ell_P}^3} \qquad \text{and} \qquad T_{M5} = \frac{1}{2^{7/3}\pi^{1/3}\,{\ell_P}^6} \ .$$

Now, the p-branes obtained in the previous lecture can be seen to have a tension, when understood as Dp-branes,

$$T_{Dp} = \frac{1}{(2\pi)^p\, g_s\, {\ell_s}^{p+1}} \ .$$

It is immediate to see that $T_{M2} = T_{D2}$, since $\ell_P \sim g_s^{1/3}\,\ell_s$ (the equal sign requires thorough computations).

Compactification along a world-volume or transverse direction gives:

- for the M2-brane, the F1-string and the D2-brane, and
- for the M5-brane, the D4-brane and the NS5-brane.

S-duality

Recall the Lagrangian of type IIB supergravity,

$$S_{IIB} = \frac{1}{16\pi G_{10}^N} \int d^{10}x \sqrt{-g}\, e^{-2\Phi} \left[\left(R + 4\,\partial_\mu \Phi\, \partial^\mu \Phi - \frac{1}{12} {H_{[3]}}^2 \right) - \frac{1}{2}\,\partial_\mu \chi\, \partial^\mu \chi - \frac{1}{12} {F_{[3]}}^2 - \frac{1}{240} {F_{[5]}}^2 \right] + \frac{1}{16\pi G_{10}^N} \int C_{[4]} \wedge F_{[3]} \wedge H_{[3]}$$

supplemented by the additional on-shell constraint $F_{[5]} = {}^\star F_{[5]}$.

The scalars and the 2-form potentials come in pairs. The equations of motion of type IIB supergravity, indeed, are invariant under a $SL(2,\mathbb{R})$ symmetry:

If we arrange the R-R scalar and the dilaton in a complex scalar

$$\lambda := \chi + i\, e^{-\Phi} \qquad \Rightarrow \qquad \lambda \to \frac{a\,\lambda + b}{c\,\lambda + d}\,,$$

where the real parameters are such that $a\,d - b\,c = 1$. Something similar happens with the NS-NS $B_{[2]}$ and the R-R $C_{[2]}$ 2-form potentials:

S-duality

They also transform according to:

$$\begin{pmatrix} B_{[2]} \\ C_{[2]} \end{pmatrix} \rightarrow \begin{pmatrix} d\, B_{[2]} - c\, C_{[2]} \\ a\, C_{[2]} - b\, B_{[2]} \end{pmatrix} ,$$

with an element of $SL(2,\mathbb{R})$.

Since $B_{[2]}$ couples to the fundamental string and the corresponding charge is *quantized*, $SL(2,\mathbb{R}) \rightarrow SL(2,\mathbb{Z})$.

Consider $\chi = 0$, and the $SL(2,\mathbb{Z})$ transformation

$$g_s \rightarrow \frac{1}{g_s} \qquad B_{[2]} \rightarrow C_{[2]} \qquad C_{[2]} \rightarrow -B_{[2]} \ .$$

This particular transformation is often referred to as S-duality.

It is a non-perturbative duality, since it exchanges weak and strong coupling.

However, instead of exchanging at the same time electric and magnetic d.o.f., it exchanges NS-NS and R-R fields, both electric.

S-duality

We argued that there are no R-R-charged states in the perturbative string spectrum. We see that, if S-duality is a symmetry of type IIB string theory, then there must be non-perturbative objects carrying RR-charge.

Notice that $SL(2,\mathbb{Z})$ is a duality relating different regimes of the same theory.

A general transformation maps the fundamental string into a general (p,q) string. It must be possible to quantize it and reproduce the type IIB theory.

The (p,q) string is *solitonic*, its tension $T_{(p,q)} \simeq 1/g_s$. This gives the string scale of the dual theory: thus α' should not be invariant under S-duality.

Since G_{10}^N is invariant, and $G_{10}^N \sim g_s^2\,\alpha'^4$, we see that $\alpha' \to g_s\,\alpha'$.

Given that type IIB supergravity is $SL(2,\mathbb{Z})$ invariant, it should be possible to write down it using explicitly $SL(2,\mathbb{Z})$ covariant degrees of freedom:

S-duality

In the Einstein frame,

$$S^E_{IIB} = \frac{1}{16\pi G^N_{10}} \int d^{10}x \sqrt{-g} \left[\left(R - \frac{\partial_\mu \bar{\lambda}\, \partial^\mu \lambda}{2\,(\mathrm{Im}\lambda)^2} - \frac{\mathcal{M}_{ij}}{2} F^i_{[3]} \cdot F^j_{[3]} \right) - \frac{1}{240} \widetilde{F}_{[5]}{}^2 \right] + \frac{\epsilon_{ij}}{32\pi G^N_{10}} \int C_{[4]} \wedge F^i_{[3]} \wedge F^j_{[3]} \,,$$

where

$$F^i_{[3]} := \begin{pmatrix} H_{[3]} \\ F_{[3]} \end{pmatrix} \qquad \mathcal{M}_{ij} := \frac{1}{\mathrm{Im}\lambda} \begin{pmatrix} |\lambda|^2 & -\mathrm{Re}\lambda \\ -\mathrm{Re}\lambda & 1 \end{pmatrix} .$$

This is invariant under

$$\lambda' = \frac{a\,\lambda + b}{c\,\lambda + d} \qquad F^{i'}_{[3]} = \Lambda^i{}_j\, F^j_{[3]} \qquad \widetilde{F}'_{[5]} = \widetilde{F}_{[5]} \qquad g^{E'}_{\mu\nu} = g^E_{\mu\nu}$$

where

$$\Lambda^i{}_j = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \qquad \mathcal{M}'_{ij} = (\Lambda^{-1})^t\, \mathcal{M}_{ij}\, \Lambda^{-1}$$

T-duality

The world-sheet action in the conformal gauge including all background fields corresponding to the NS-NS sector of the closed string reads

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left[\sqrt{-h}\left(h^{\alpha\beta}\, g_{\mu\nu}\, \partial_\alpha x^\mu \partial_\beta x^\nu - \alpha'\, \Phi\, R^{(2)}\right) - \epsilon^{\alpha\beta}\, B_{\mu\nu}\, \partial_\alpha x^\mu \partial_\beta x^\nu\right] .$$

A few comments are in order:

- When the Φ is constant, the second term captures the topology of the world-sheet (the Euler characteristic is determined by the genus).

  The genus is nothing but the number of string loops!

- The $B_{\mu\nu}$ term is the pull-back of $B_{[2]}$.
- Notice that the tension of the string equals its charge under $B_{[2]}$.

Now, consider a circular coordinate, say x^9, and background fields that do not depend on it.

T-duality

The action can be written in terms of a Lagrange multiplier, $\widetilde{x}^9$, as

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left[\sqrt{-h}\, h^{\alpha\beta} \left(g_{\mu\nu}\, \partial_\alpha x^\mu \partial_\beta x^\nu + 2 g_{\mu 9}\, V_\alpha \partial_\beta x^\mu + g_{99}\, V_\alpha V_\beta \right) \right.$$

$$\left. - \epsilon^{\alpha\beta} \left(B_{\mu\nu}\, \partial_\alpha x^\mu \partial_\beta x^\nu - B_{\mu 9}\, V_\alpha \partial_\beta x^\mu \right) - \widetilde{x}^9\, \epsilon^{\alpha\beta}\, \partial_\alpha V_\beta - \alpha' \sqrt{-h}\, \Phi\, R^{(2)} \right] .$$

Indeed, the $\widetilde{x}^9$ equation of motion is

$$\epsilon^{\alpha\beta}\, \partial_\alpha V_\beta = 0 \qquad \Rightarrow \qquad V_\beta = \partial_\beta x^9 .$$

Substituting this into the action takes as back to the previous one.

On the other hand, using the V_α equations leads to a dual action with

$$\widetilde{g}_{99} = \frac{1}{g_{99}} \qquad \widetilde{g}_{9\mu} = \frac{B_{9\mu}}{g_{99}} \qquad \widetilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{B_{9\mu}\, B_{9\nu} - g_{9\mu}\, g_{9\nu}}{g_{99}}$$

$$\widetilde{B}_{9\mu} = -\widetilde{B}_{\mu 9} = \frac{g_{9\mu}}{g_{99}} \qquad \widetilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{g_{9\mu}\, B_{9\nu} - B_{9\mu}\, g_{9\nu}}{g_{99}} .$$

T-duality

A full understanding of T-duality would require a microscopic analysis (that Javier will do). There you will see that, in terms of the left and right moving momenta, the T-duality transformation becomes:

$$p_L^9 \leftrightarrow p_L^9 \qquad p_R^9 \leftrightarrow -p_R^9 \qquad \tilde{\alpha}_n \leftrightarrow -\tilde{\alpha}_n \, .$$

In other words,

$$x^9 = x_L^9 + x_R^9 \qquad \leftrightarrow \qquad x'^9 = x_L^9 - x_R^9 \, .$$

Because of the world-sheet supersymmetry, the fermionic superpartner ψ^9 also has to transform under T-duality, as $\psi_L^9 \leftrightarrow \psi_L^9$ and $\psi_R^9 \leftrightarrow -\psi_R^9$.

T-duality acts like a space-time parity reversal restricted to the right moving modes: the chirality of the corresponding Ramond ground state changes.

T-duality maps type IIA and type IIB string theories among themselves!

This means that both theories compactified on a circle are equivalent at the perturbative level (it extends to a non-perturbative symmetry).

T-duality

It is possible to see that, under T-duality, the type IIA and type IIB coupling constants are related by

$$\widetilde{g}_s = g_s \frac{\sqrt{\alpha'}}{R} ,$$

where R is the radius of the circle along x^9 (that the string winds). Then,

$$g_{99} = \frac{R^2}{\alpha'} \qquad \text{and} \qquad \widetilde{g}_{99} = \frac{\widetilde{R}^2}{\alpha'} = \frac{1}{g_{99}} \quad \Rightarrow \quad \widetilde{\Phi} = \Phi - \frac{1}{2}\, g_{99} .$$

Big circles are T-dual to small circles! What about the R-R sector fields?

Since T-duality is a space-time parity reversal restricted to the right moving modes, it transform type IIA R-R tensor fields into type IIB ones and viceversa:

$$\widetilde{C}_9 = C \qquad \widetilde{C}_\mu = C_{\mu 9} \qquad \widetilde{C}_{\mu\nu 9} = C_{\mu\nu} \qquad \widetilde{C}_{\mu\nu\lambda} = C_{\mu\nu\lambda 9} .$$

(these formulas are strictly valid for trivial NS-NS backgrounds).

Correspondingly, type IIA D-branes map into type IIB ones and viceversa.