# Notes on String Theory

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# Contents


**1 Strings from gauge theories** **3**

1.1 Introduction . . . 3

1.2 The dual resonance model . . . 3

1.3 Regge trajectories from String Theory . . . 7

1.4 String theory from the large $N_c$ counting of gauge theories . . . 8

**2 Open and Closed String Amplitudes** **13**

2.1 Closed string interactions . . . 13

2.2 $S$ matrix and Scattering Amplitudes . . . 18

2.2.1 $m$-point amplitudes for closed strings. . . . 18

2.2.2 4-point tachyon amplitude . . . 21

2.3 Open string scattering . . . 24

**3 Gauge theory from string theory** **29**

3.1 Gauge theory from open strings . . . 29

3.2 Gauge theory from closed strings . . . 30

3.2.1 The Kaluza Klein reduction . . . 30

3.2.2 Closed strings on a circle . . . 31

3.2.3 Massless spectrum . . . 33

3.3 T- duality . . . 33

3.3.1 Full spectrum: T-duality for closed strings . . . 34

3.3.2 Enhanced gauge symmetry: . . . 34

**4 D-branes** **35**

4.1 T-duality for open strings and D-branes . . . 35

4.1.1 D branes . . . 36

4.1.2 T-duality with Wilson Lines . . . 36

4.1.3 T-duality for type II superstrings . . . 37

4.2 Dynamics of $D-branes$ . . . 39

4.2.1 Born-Infeld action . . . . . . . . . . . . . . . . . . . . . . . . . . . . 40

# Chapter 1

# Strings from gauge theories

## 1.1 Introduction

We can safely trace the origins of strings to the early 50's, where lots of scattering experiments were performed involving mesons and baryons. The evidence of a parton substructure, together with the impossibility to obtain the quarks as a individual final states lead to the conclusion that confinement is a built in property of non abelian gauge theories. Later on, in the 70's, many theoretical efforts were devoted to prove this statement, which so far remains elusive. The suspect is that the strength of the interaction is such that brings the problem outside the reach of usual perturbative methods in QFT. An utmost brilliant idea was to use the number of colors $N_c$ to build a perturbation expansion parameter proportional to its inverse. In this chapter we will review some of the ideas that lead to the radical proposal that, at least at low energies, the degrees of freedom of QCD might be closer to those of strings than to point particles.

## 1.2 The dual resonance model

*1.2.0.1 Regge trajectory*

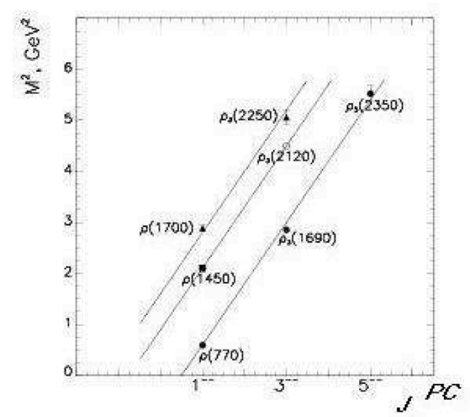
The main assumption in the *dual resonance model* is that hadrons interact through the formation of intermediate states (resonances).

In the '50',s mesons and baryons were discovered to have many excited states of arbitrarily high spin $J$ and masses along the so called "leading Regge trajectory" [1]

$$J = \alpha_0 + \alpha' M^2 \tag{1.1}$$

with $\alpha' \sim 1(\mathrm{GeV})^{-2}$.

[1] there also appeared to be subleading Regge trajectories $M^2 \sim (J-n)/\alpha',\ n = 1, 2, \ldots$




Indeed these appeared as resonances in the elastic scattering amplitudes of two incoming mesons of momenta $p_1, p_2$ into two mesons of outgoing momenta $-p_3, -p_4$. Consider the Mandelstam invariants

$$s = -(p_1 + p_2)^2 \quad ; \quad t = -(p_2 + p_3)^2 \quad ; \quad u = -(p_1 + p_3)^2 \,. \tag{1.2}$$

These are not independent, but satisfy $s + t + u = -\sum_{i=1}^{4} M_i^2$. Amplitudes are therefore functions $A^{(4)}(s,t)$ with poles in the $s$ and/or the $t$ channel. Since strong interactions preserve flavor symmetry, each external meson can be assigned a flavor matrix $\lambda_i$ and the total amplitude must be proportional to $\mathrm{tr}(\lambda_1\lambda_2\lambda_3\lambda_4)$ which is invariant under cyclic permutations $1234 \to 2341$. Bose symmetry then says that the corresponding amplitude should be cyclically symmetric $p_1p_2p_3p_4 \to p_2p_3p_4p_1$, or $A^{(4)}(s,t) = A^{(4)}(t,s)$.

In field theory this "crossing" symmetry is implemented by explicitely adding up the two channels. ¿Would it be possible to have this symmetry by just considering either the $s$ or the $t$ channel alone? Certainly an $s$-channel amplitude mediated by spin $J$ particles would be of the form [2]

$$A(s,t) = \sum_J \frac{g_J^2(-t)^J}{s - M_j^2} \tag{1.3}$$

[2] In field theory, the exchange of a spin $J$ particle $\sigma^{\mu_1\cdots\mu_J}$ of mass $M$, in a scalar 4 pony function comes from an interaction vertex with $J$ derivatives $\phi\sigma^{\mu_1\cdots\mu_J}\partial_{\mu_1}...\partial_{\mu_J}\phi$. Hence it leads to a contribution with $p^J$ powers of momenta, $A^{(4)}(s) \sim t^J/(s - M^2) + ....$

If the sum truncates to a finite $J$ we observe

- a severe power law divergence at high energy $t \to \infty$.
- no poles for finite values of $t$, hence no symmetry under $s \leftrightarrow t$-

But, could this be remedied by letting the sum run over unbounded spins? $J = 1, 2, 3, ....$

*1.2.0.2 Veneziano amplitude*

In 1968 Veneziano proposed the following heuristic answer

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-(\alpha(s)+\alpha(t)))} \tag{1.4}$$

with $\alpha(s) = \alpha(0) + \alpha' s$.

Euler Gamma function has poles in the negative real axis at integer values $\alpha(s) = n$ with residue

$$\begin{aligned}\Gamma(-\alpha(s)) &= \frac{\Gamma(-\alpha(s)+1)}{-\alpha(s)} \\ &= \frac{\Gamma(-\alpha(s)+n+1)}{-\alpha(s)(-\alpha(s)+1)(-\alpha(s)+2)...(-\alpha(s)+(n-1))(-\alpha(s)+n)} \\ &\stackrel{\alpha(s)\to n}{\longrightarrow} \frac{(-1)^n}{n!}\frac{1}{-\alpha(s)+n}\end{aligned} \tag{1.5}$$

Hence, at fixed $t$, the amplitude has infinitely many poles at $s \in (0, \infty)$ for $\alpha(s) = \alpha(0) + s\alpha' = n$ or

$$s = \frac{n - \alpha(0)}{\alpha'} = M_n^2 \tag{1.6}$$

with residue

$$\begin{aligned}A^{(4)}(s,t) &\stackrel{\alpha(s)\to n}{=} \frac{(-1)^n}{n!}\frac{\Gamma(-\alpha(t))}{\Gamma(-n-\alpha(t))}\frac{1}{\alpha(s)-n} \\ &= \frac{(\alpha(t)+1)(\alpha(t)+2)...(\alpha(t)+n))}{n!}\frac{1}{\alpha(s)-n}\end{aligned} \tag{1.7}$$

For $\alpha(t) = \alpha(0)' + \alpha' t$ a linear function of t, we see that the residue is a polynomial in $t$ of order $n$ much like the numerator in (1.12) with $J = n$. Since

$$t = -(p_2 + p_3)^2 = M_2^2 + M_3^2 + 2E_2E_3 - 2|p_2||p_3|\cos(\theta) \tag{1.8}$$

we see that it contains the power $(\cos(\theta))^n$. This implies that the $n$th resonance in the $s$ channel consist of particles of mass $M_n^2 = (n - \alpha_0)/\alpha'$ and with maximal angular momentum $J = n$. This implies the Veneziano formula entails the so called, "leading Regge trajectory"

$$M^2 = \frac{J - \alpha_0}{\alpha'}\,. \tag{1.9}$$

The denominator in (1.4) is essential to avoid double poles at $\alpha(s) = n$ and $\alpha(t) = m$, since QFT only permits simple poles in a tree level amplitude.

#### 1.2.0.3 *Truncating*

In summary, the full amplitude can be expressed as an infinite sum

$$A^{(4)}(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(t)+1)(\alpha(t)+2)...(\alpha(t)+n))}{n!} \frac{1}{\alpha(s)-n} \qquad s\text{-channel} \tag{1.10}$$

The original expression is manifestly symmetric under $s \leftrightarrow t$ while this one isn't. Therefore we can also write another expression for the $t - channel$

$$A^{(4)}(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)(\alpha(s)+2)...(\alpha(s)+n))}{n!} \frac{1}{\alpha(t)-n} \qquad t\text{-channel} \tag{1.11}$$

For large $s$ or $t$ the leading contribution comes from the exchanged particle of highest spin

$$A^{(4)}(s,t) = -\sum_{J=0}^{\infty} \frac{(-t)^J}{s-M_J^2} + ... \qquad s\text{-channel} \tag{1.12}$$

$$A^{(4)}(s,t) = -\sum_{J=0}^{\infty} \frac{(-s)^J}{t-M_J^2} + ... \qquad t\text{-channel} \tag{1.13}$$

Notice that any truncation to a finite sum yields a power divergence in $t^J(s^J)$ for the $s(t)$-channel. However summing up infinite terms does two jobs. On one hand it restores duality. On the other it softens the ultraviolet behaviour. In this sense we must distinguish among two possible asymptotics

#### 1.2.0.4 *High energy behaviour of the Veneziano amplitude*

Let us take the following incoming and outgoing momenta

$$p_1 = \frac{\sqrt{s}}{2}(1,1,0,...) \quad ; \quad p_2 = \frac{\sqrt{s}}{2}(1,-1,0,...)$$
$$-p_3 = \frac{\sqrt{s}}{2}(1,-\cos\theta,-\sin\theta,...) \quad ; \quad -p_4 = \frac{\sqrt{s}}{2}(1,\cos\theta,\sin\theta,...) \tag{1.14}$$

Then

$$\begin{aligned} t &= -(p_2+p_3)^2 = \frac{s}{4}\left(-0^2 + (1-\cos\theta)^2 + \sin^2\theta\right) \\ &= \frac{s}{2}(1-\cos\theta) \end{aligned} \tag{1.15}$$

- *Large $s$ at fixed $t$.* From (1.15) we see that this is the same as small relative angle $\theta \to 0$. Using asymptotic Stirling's formula

$$\Gamma(u) \overset{u\to\infty}{\to} u^{u-1/2}e^{-u} \tag{1.16}$$

we have

$$\begin{aligned} \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(s))}{\Gamma(-\alpha(s)-\alpha(t))} \quad &\overset{\alpha(s)\to\infty}{\longrightarrow} \sim \frac{\Gamma(-\alpha(t))(-\alpha(s))^{-\alpha(s)-1/2}e^{\alpha(s)}}{(-\alpha(s)-\alpha(t))^{-\alpha(s)-\alpha(t)-1/2}e^{\alpha(s)}} \\ &\to \quad \sim \Gamma(-\alpha(t))(-\alpha(s))^{\alpha(t)} \end{aligned} \tag{1.17}$$

Hence, for a linear Regger trajectory, $\alpha(s) = \alpha_0 + \alpha' s$, the asymptotic behaviour for large $s$ fixed $t$ is a power law

$$A(s,t) \sim s^{\alpha(t)} \tag{1.18}$$

In the region of elastic scattering $s > 0$ but $t < 0$, so $\alpha(t) < 0$ and we have a soft behaviour, that contrasts with the diverging $s^J$ contribution in (1.13).

- *Large $s$ and $t$ at fixed angle*

  From (1.15) we see that we can have fixed angle if we take simultaneously $s, t \to \infty$. Doing the same as before with $\alpha(t) = \alpha(s) f(\cos(\theta))$

$$\begin{aligned} A(s,t) &\to \frac{(-\alpha(t))^{-\alpha(t)-1/2} e^{\alpha(t)} (-\alpha(s))^{-\alpha(s)-1/2} e^{\alpha(s)}}{(-\alpha(s) - \alpha(t))^{-\alpha(s)-\alpha(t)-1/2} e^{\alpha(s)+\alpha(t)}} \\ &\to \frac{(-\alpha(s))^{-\alpha(t)-\alpha(s)-1}}{(-\alpha(s))^{-\alpha(t)-\alpha(s)-1/2}} g(\cos(\theta))^{-\alpha(s)} \\ &\to P(s) g(\cos(\theta))^{-\alpha(s)} \end{aligned} \tag{1.19}$$

  with $P(s)$ some polynomial of $s$. Hence, this channel is even softer, being actually exponentially decaying.

## 1.3 Regge trajectories from String Theory

The first hint at a string theoretical origin of the relation (1.1) was given by Nambu. Consider a massless and spineless quar-antiquark pair connected by a rigid rod of length $2r_0$. This rod, or rigid string, mimics the flux tube that connects both quarks binding them into a meson, and it has a tension, or energy per unit length $\sigma$ [3] For a given length, the largest achievable angular momentum $J$ occurs when the ends of the string move with the velocity of light. In these circumstances, the linear velocity of a point along the string at a distance $r$ from the centre will be $v(r) = r/r_0$. The total mass of the system is then given by

$$M = 2\int_0^{r_0} \frac{\sigma}{\sqrt{1 - v(r)^2}} dr = 2\sigma \int_0^{r_0} \frac{dr}{\sqrt{1 - (r/r_0)^2}} = \sigma \pi r_0 \tag{1.20}$$

On the other hand, the orbital angular momentum of the string is

$$J = 2\int_0^{r_0} \frac{\sigma r v}{\sqrt{1 - v(r)^2}} dr = \frac{2\sigma}{r_0} \int_0^{r_0} \frac{r^2 dr}{\sqrt{1 - (r/r_0)^2}} = \sigma \frac{\pi}{2} r_0^2 \,. \tag{1.21}$$

From these two relations we deduce that

$$J = \frac{M^2}{2\pi\sigma} \tag{1.22}$$

[3] Notice that the string tension is a force, *i.e.* a (negative) pressure integrated over the transverse section of the string. This has the same units as an energy per unit length.

which mimics precisely a Regge trajectory with $\alpha(0) = 0$ and

$$\alpha' = \frac{1}{2\pi\sigma}\,. \tag{1.23}$$

If phenomenologically we know that $\alpha' \sim 0.9$ GeV$^{-2}$, this implies for the present model a string tension $\sigma = 0.18$ GeV$^2$ (in natural units).

This constant tension, or linear energy density, suggests that the inter-quark binding force stems from a (confining) potential of the form

$$V(r) = \sigma r \tag{1.24}$$

## 1.4 String theory from the large $N_c$ counting of gauge theories

The last argument that we shall address in this speculative approach that intends to highlight the relation among gauge theories and string theory stems from the perturbative expansion of a Yang Mills theory. Starting from the lagrangian of QCD and doing the following replacements

$$A = \hat{A}/g_{YM} \quad ; \quad \psi = \sqrt{N}\hat{\psi} \quad ; \quad g_{YM} = \sqrt{\lambda/N} \tag{1.25}$$

we arrive at the form (after omitting all the hats)

$$\mathcal{L}_{QCD} = N\left[-\frac{1}{2\lambda}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \sum_{k=1}^{N_f}\bar{\psi}_k(i\not{D} - m_k)\psi\right] \tag{1.26}$$

where

$$\not{D}_\mu = \partial_\mu + iA_\mu \tag{1.27}$$

$$A_\mu = A_\mu^A T^A \quad ; \quad \mathrm{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB}\,. \tag{1.28}$$

with $T^A$ traceless hermitian generators of $SU(N)$, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu]\,. \tag{1.29}$$

This form of the lagrangian is very well suited to do the analysis of the perturbative expansion for large $N$. In principle one should start by obtaining the propagators

$$\langle\psi^a(x)\bar{\psi}^b(y)\rangle = \frac{\delta^{ab}}{N}S(x-y), \tag{1.30}$$

$$\langle A_\mu^A(x)A_\mu^B(y)\rangle = \frac{\lambda}{N}\delta^{AB}D_{\mu\nu}(x-y)\,. \tag{1.31}$$

The $\delta^{ab}$ in the quark propagator means that color is conserved along the line. It is desirable to treat the gluon fields in a similar way, and this can be achieved thanks to a trick due originally to ´t Hooft. Instead of treating the gluon field as a field with a single adjoint index $A^A_\mu$, it is preferably to treat is as an $N \times N$ matrix with two indices in the $N$ and $\bar{N}$ representations of $SU(N)$

$$A^a_{\mu b} = A^A_\mu (T^A)^a{}_b\,. \tag{1.32}$$

The gluon propagator for these fields can be rewritten as [4]

$$\langle A^a_{\mu b}(x) A^c_{\nu d}(y)\rangle = \frac{\lambda}{N} D_{\mu\nu}(x-y)\left(\frac{1}{2}\delta^a_d\delta^c_b - \frac{1}{2N}\delta^a_b\delta^c_d\right) \tag{1.34}$$

Now we can express both propagators using lines that conserve colour

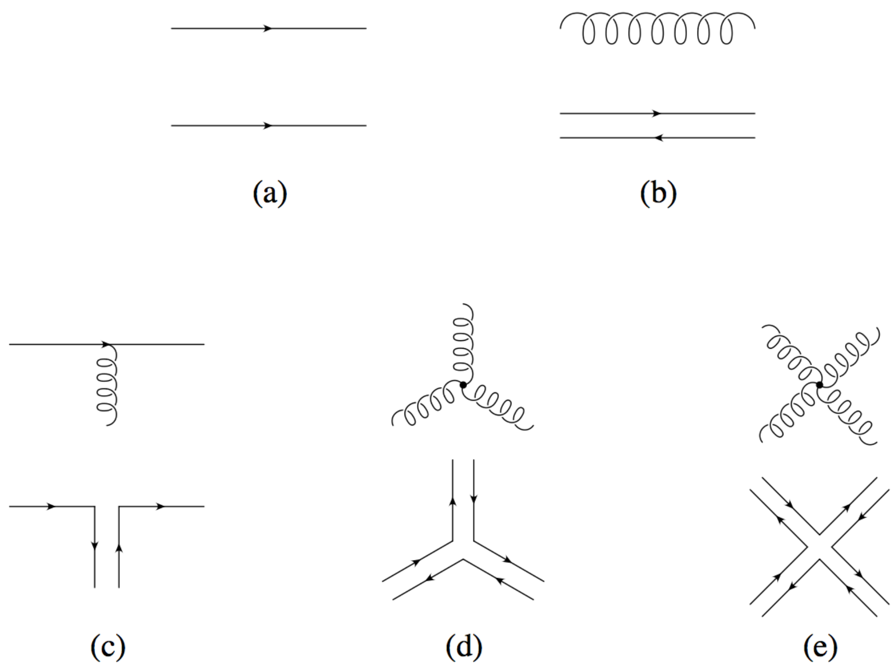


Figure 1.1: Feynman rules in double line notation

As one can see in figure (**??**) the quark propagator has colour indices $a = b$ attached at its ends. Similarly, a gluon propagator can be expressed as a double line with indices $a = d$ and $c = b$. In figures $(c)$, $(d)$ and $(e)$ we see the interaction vertices in the double line notation. They all come from pieces out of $\mathrm{Tr}F^2$. For example the triple vertex in $(d)$ comes with

$$(N/\lambda^2)\mathrm{Tr}\ A_\mu A_\nu \partial_\mu A_\nu = (N/\lambda) A^a_{\mu b} A^b_{\nu c} \partial_\mu A^c_{\nu a}$$

From every Feynman graph in the original theory we obtain a new graph in the double line notation. A double line graph looks then as a surface tesellated with polygons which

[4] the identity

$$(T^A)^a{}_b (T^A)^c{}_d = \frac{1}{2}\delta^a_d\delta^c_b - \frac{1}{2N}\delta^a_b\delta^c_d \tag{1.33}$$

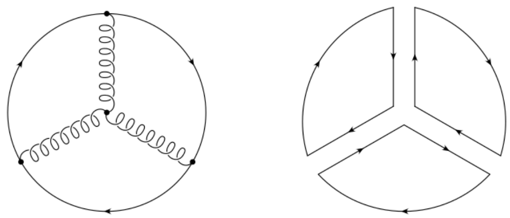


Figure 1.2: 3 loop vacuum diagram in double line notation

are glued adjacently along the double lines. Since each line is either an $N$ or a $\bar{N}$ we can assign an arrow to each on and glue them with opposite senses (as a gluon transforms in the $N \times \bar{N}$). The each polygon corresponds to an orientable surface tesellation. Color index loops correspond to the edges of a polygon.

When counting the power of $N$ that accompanies a given Feynman diagram one notices the following rules

- Each quark propagator carries a factor of $N$.
- Each gluon propagator carries a factor of $\lambda/N$.
- Each gluon vertex yields a factor of $N/\lambda$.
- Each (quark or gluon) vertex yields a factor of $N$.
- for each closed loop, there is an additional factor of $N$ from the trace over color space.

In this ways, one can check a few examples. Consider a diagram with only gluons. Hence we see that a generic diagram will come with a factor of

$$\left(\frac{N}{\lambda}\right)^{\text{number of vertices}} \times \left(\frac{\lambda}{N}\right)^{\text{number of lines}} \times N^{\text{number of loops}} \tag{1.35}$$

Consider the two examples in the figure. The one on the left hand side comes with a factor of $\lambda^3 N^2$, whereas the one on the right hand side comes only with $\lambda^2 N^0$. The very smart observation by ‘t Hooft is the fact that, whereas the first diagram can be drawn on a sphere the second one need a torus, in order to avoid self crossing of lines. In other word the exponent of $N$ yields precisely the genus of a Riemann surface by properly interpreting the loops as faces, lines as edges and vertices as such

$$N^{\text{vertices}-\text{lines}+\text{loops}} = N^{V-E+F} = N^{\chi} = N^{2-3h}$$

where

$$\chi = 2 - 2h \tag{1.36}$$

of $SU(N)$ has been used.

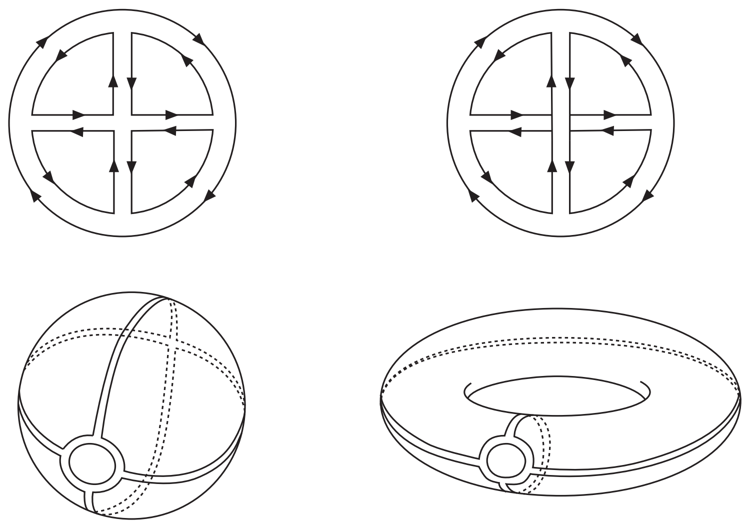


Figure 1.3: Four loop vacuum graph in double line notation. Planar an non planar contributions.

is the genus of a Riemann surface with $g$ handles. A sphere has $h = 0$ hence $\chi = 2$ whereas a torus has $h = 1$ and so $\chi = 0$.

Actually by plain around a little bit more one can show that this formula generalizes to the presence of quark loops by enhancing the Euler characteristic as

$$\chi = 2 - 2h - b \tag{1.37}$$

with $b$ the number of boundaries.

In summary, topology organizes the diagram in sets of given order in powers of $N^{\chi}$. On the contrary, given a fixed topology, the order of $\lambda$ is not fixed at all. Therefore, we may reorganize the path integral as a sum of subsets of (connected) diagrams with given topology

$$\ln Z = \sum_{h=0}^{\infty} (1/N)^{2h-2} \mathcal{F}_h(\lambda) \tag{1.38}$$

where each $\mathcal{F}_h(\lambda) = \sum_{l=0}^{\infty} \lambda^l c_{h,l}$ is itself a sum of diagrams.

Notice that we have put $N$ in such a way that makes obvious that, for large $N \gg 1$ it is a perturbative expansion in powers of $1/N$. In order to be also perturbative in $\lambda$ we need to set $\lambda = g_{YM}^2 N$ fixed and small. This requires $g_{YM} \ll 1$ as $N \gg 1$. The strict limit $N \to \infty$ and $g_{YM}^2 \to 0$ with $\lambda$ fixed is called ‘t Hooft limit, in which the path integral is well approximated by the planar diagrams. This includes a subset of all loop diagrams, and has way more information than the tree level or semiclasical approximation.

Moreover, the coupling constant $\lambda$ acts as a kind of ”chemical potential” for edges in the triangulations. Therefore, if $\lambda$ is large, the diagrams with lots of edges become more important, and leading to a picture of really smooth surfaces filled by propagators. This loos connection of gauge theories in the large $N$ limit with string theory has only been proven rigorously in 2 dimensions.

# Chapter 2

# Open and Closed String Amplitudes

## 2.1 Closed string interactions

2.1.0.5 *String perturbation theory*

Let us for a moment take seriously the motivation to study string theory as a means of taming the full perturbative expansion of QCD in the large N limit. As we saw previously, the suspect is that the partition function resembles that of a theory of Riemann surfaces weighted by $(1/N)^{2g-2}$ where $g$ is the genus or, equivalently, the number of handles.

Therefore we are naturally lead to study a theory of surfaces in space-time and these are none other than world sheets of strings. The natural way to setup a quantum mechanics of such strings is to devise a classical action for the world sheets, and the easiest solution is given by the Nambu-Goto action. However this leads to non-linear equations of motion and therefore an elegant tweak was provided by Polyakov, who suggested to decouple the intrinsic metric from the induced one.

$$S_{\text{Polyakov}} = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-g}\, g^{ab} \partial_a X \cdot \partial_b X \tag{2.1}$$

where $X^I, I = 1, ...., D+1$ is a set of ambient coordinates. With $g_{ab}(X)$ a function of $X^I$, this is a 2-dimensional non-linear sigma model over some 2 dimensional surface $\Sigma$.

Of course, we would like to sum over world sheet Riemann surfaces, and even more, weight them differently. This can be achieved by a slight enhancement of the previous action

$$\begin{aligned} S_{\text{string}} &= S_{\text{Polyakov}} + \phi\chi \\ &= S_{\text{Polyakov}} + \frac{\phi}{4\pi} \int d^2\sigma \sqrt{g} R \end{aligned} \tag{2.2}$$

where the added term is a topological invariant (a total derivative in two dimensions), and $\chi = (2-2g)$ for a genus $g$ Riemann surface. In 2 dimensions the Einstein term does not make gravity dynamical, as it can be locally gauged away. Now the path integral looks as follows

$$\sum_{\substack{\text{topologies}\\ \text{metrics}}} e^{-S_{\text{string}}} \sim \sum_{\text{topologies}} (e^{\phi})^{2g-2} \int (\mathcal{D}X) \frac{(\mathcal{D}g)}{Vol} e^{-S_{\text{Polyakov}}[g,x]} \tag{2.3}$$

For small string coupling constant $g_s \equiv e^{\phi}$ the genus expansion is a perturbative expansion and the tree level approximation is given by the sphere. We see that $e^{\phi}$ plays the role of $1/N$ in gauge theories. We have replaced Feynman diagrams by Riemann surfaces.

The factor $Vol$ in the previous expression reminds us of the fact that there is a huge gauge redundancy in the path integral. As in ordinary gauge theories we are then instructed to include in the path integral only gauge inequivalent configurations. The resulting manifold is much smaller, and this is roughly what is meant by formal expression $\mathcal{D}(g)/Vol$.

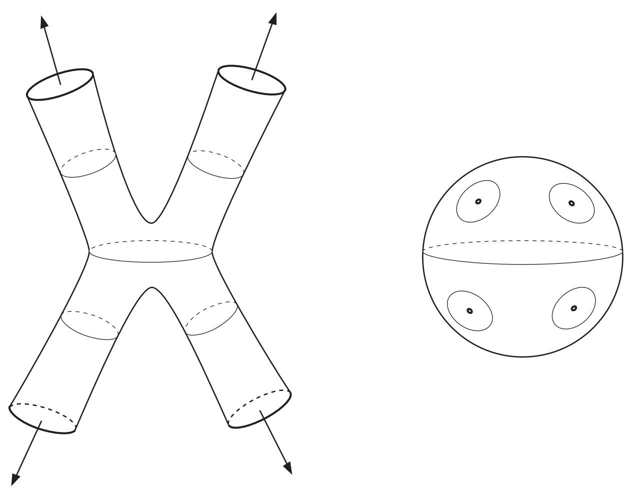


Figure 2.1: Four string scattering at tree level and its related conformal transformation into a Riemann sphere with punctures.

#### 2.1.0.6 *Gauge fixing*

The two gauge symmetries we have at hand are

- General coordinate invariance

$$\sigma^{\alpha} \to \sigma^{\alpha} + v^{\alpha}(\sigma) \quad \Rightarrow \quad g_{ab} = g_{\alpha\beta} + \nabla_{\alpha} v_{\beta} + \nabla_{\beta} v_{\alpha} \tag{2.4}$$

  In two dimensions, any metric can be brought locally to the so called conformal gauge $g_{ab} = e^{\phi}\delta_{ab}$

- local Weyl symmetry

$$g_{ab} \to e^{\varphi(\sigma^a)} g_{ab} \tag{2.5}$$

  This "classical" symmetry of the action is only also a true "quantum symmetry" of the full path integral in $D = 26$ dimensions for the bosonic string, or in $D = 10$ for the superstring.s

String amplitudes are obtained by summing over all world-sheets bounded by initial and final curves. For asymptotic states we consider these boundaries to extend the world sheet to past (incoming) and future (outgoing) infinity (see fig. 2.1).

Each leg, when taken to infinity should be prepared in a well defined state of spacetime momentum $p_i$ and quantum numbers $\lambda_i$. The key observation is that the conformal symmetry allows to transform surfaces with boundaries at infinity into punctured Riemann surfaces. The incoming and outgoing strings are conformally mapped onto points (the punctures) of the two dimensional surface.

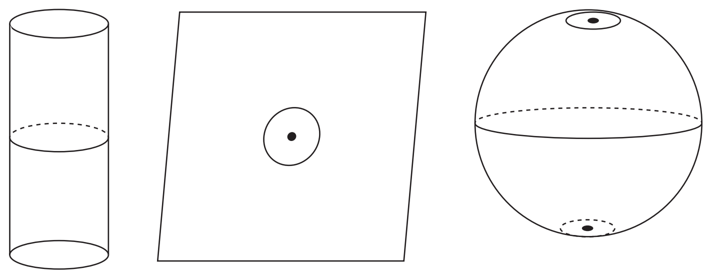


Figure 2.2: *String propagator in different coordinates.*

#### *2.1.0.7 Conformal mapping of amplitudes*

For example, consider the simplest case of a genus zero world-sheet with one incoming and one outgoing string. This can be described by an infinitely long cylinder like the one shown in fig. (2.2). On the left we have the cylinder representing a world sheet propagating a closed string from past infinity to future infinity. The euclidean world sheet metric can be taken as $ds^2 = d\tau^2 + d\sigma^2$ with $\tau \in (-\infty, +\infty)$ and $\sigma \in (0, 2\pi)$. With $\omega = \sigma + i\tau$, $\bar{\omega} = \sigma - i\tau$ we have mapped the cylinder onto the horizontal strip in the complex plane of width $2\pi$, and $ds^2 = d\omega d\bar{\omega}$. Now doing $\omega = i \log z$ *i.e.* $z = e^{-i\omega}$ we get

$$ds^2 = d\sigma^2 + d\tau^2 = d\omega d\bar{\omega} = \frac{dz d\bar{z}}{z\bar{z}} \tag{2.6}$$

Finally, a conformal transformation $ds^2 \to ds'^2 = z\bar{z} ds^2 = dz d\bar{z}$ brings the cylinder to the complex plane with the past $\tau \to -\infty$ being now the origin $z = 0$ and the future $\tau \to +\infty$ being now at complex infinity $z \to \infty$. We may compactify the complex plane onto the Riemann sphere by doing another conformal transformation

$$ds''^2 = \frac{ds'^2}{(1 + |z|^2)^2} = \frac{dz d\bar{z}}{(1 + |z|^2)^2} \,. \tag{2.7}$$

which is invariant under the inversion $z \to 1/z$. Hence the past and future cylinder ends have been mapped to the north and south poles on the Riemann sphere.

Following this reasoning it should be clear now that for more complicated interaction, such as the one depicted in fig. 2.1, we may perform a change of coordinates that brings the diagram on the left onto a Riemann sphere with four punctures. Near each puncture we may chose coordinates $z_i$, such that $z_i = 0$ signals the point where an operator insertion injects the quantum numbers of the asymptotic state.

As we will show later, demanding Weyl invariance of the result forces the vertex operators to be *on shell*. Hence this is more an $S$-matrix computation rather than an off shell amplitude.

*2.1.0.8 Gauge fixing, residual gauge invariance*

As mentioned before, the factor $Vol$ arises because in summing over metrics and fields, there are a lot of gauge symmetries. Ones is then instructed to constrain the sum to gauge inequivalent worldsheets. Again, by a diffeomorphism and a conformal transformation we can alway bring the *euclidean* metric to a convenient gauge $g_{ab} \to \delta_{ab}$. Going over to complex coordinates $ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dzd\bar{z}$, the Polyakov action becomes

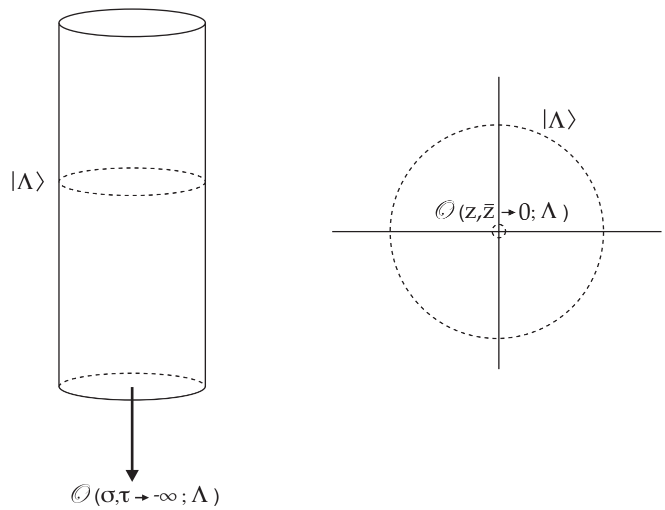
$$\begin{aligned} S_{\text{Polyakov}} &= \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{g}\, g^{ab}\, \partial_a X \cdot \partial_b X \\ &= \frac{1}{4\pi\alpha'} \int d\sigma d\tau\, \delta^{ab}\, \partial_a X \cdot \partial_b X \\ &= \frac{1}{4\pi\alpha'} \int dz d\bar{z}\; 2\, \partial_z X \cdot \partial_{\bar{z}} X \\ &= \frac{1}{2\pi\alpha'} \int dz d\bar{z}\; \partial X \cdot \bar{\partial} X \end{aligned} \tag{2.8}$$

It turns out that we have not fixed completely the gauge symmetry by writing the Polyakov action in this form. It is still invariant under *diffeomorphism that can be undone by a conformal transformation*. This is is especially easy to see in complex coordinates. Such residual gauge transformations are just analytic changes of coordinates $z \to z'(z), \bar{z} \to \bar{z}'(\bar{z})$. This important fact has two aspects worth emphasizing.

- In the neighborhood of a point, that is, locally, this residual symmetry group is infinite dimensional. It therefore allows one to use all the artillery of two dimensional CFT, where many correlators are almost algebraically determined.
- Globally the beast looks much less fierce. Indeed globally defined analytic changes of a Riemann surface are severely restricted and they typically span a finite dimensional manifold.

*2.1.0.9 State operator mapping*

The above conformal mapping allows to treat the neighborhood of any puncture like the vicinity of the origin in the complex plane. This allows to make use of the powerful techniques of 2D conformal field theory. A very important bonus of this is the remarkable fact that in 2D CFT there is a one to one correspondence between local operators (primary fields) and Hilbert space states. In essence, this isomorphisms states that creating an incoming state (past infinity) is isomorphic to placing some operator in the past, hence precisely at the puncture.




Le us see this in somewhat more detail. For instance, given a bosonic field $X(\sigma,\tau)$, for it to be well defined on the euclidean cylinder $\omega = \sigma + i\tau$ it must admit a Taylor expansion like

$$X(\sigma,\tau) = x - i\alpha' p\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha_n e^{in\omega} + \tilde{\alpha}_n e^{in\bar{\omega}}) \qquad \omega = \sigma + i\tau \tag{2.9}$$

from where, since $\tau = (\omega - \bar{\omega})/2i$ we have, for the corresponding operator

$$\partial_\omega \hat{X}(\omega) = -\sqrt{\frac{\alpha'}{2}}\sum_n \hat{\alpha}_n e^{in\omega} \qquad \hat{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}\hat{p} \tag{2.10}$$

and the same for $\partial_{\bar{\omega}}\hat{X}(\bar{\omega})$. Now doing $z = e^{-i\omega}$ we map the cylinder to the complex plane. Since $\partial X$ is a conformal field of weight 1 we have

$$\partial_z \hat{X}(z) = \left(\frac{\partial z}{\partial \omega}\right)^{-1}\partial_\omega \hat{X}(\omega) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \frac{\hat{\alpha}_m}{z^{m+1}} \tag{2.11}$$

With this we have, after defining the vacuum $|0\rangle$ as such that $\alpha_m|0\rangle = 0$ for $m \geq 0$

$$\hat{\alpha}_{-1}|0\rangle = \lim_{z\to 0}\left(\sum_m \frac{\hat{\alpha}_m}{z^{m+1}}\right)|0\rangle = i\sqrt{\frac{2}{\hat{\alpha}'}}\lim_{z\to 0}\partial\hat{X}(z)|0\rangle \tag{2.12}$$

and more generally

$$\hat{\alpha}_{-n}|0\rangle = \lim_{z\to 0}\left(\sum_m \frac{\hat{\alpha}_m}{z^{m+n}}\right)|0\rangle = i\sqrt{\frac{2}{\hat{\alpha}'}}\frac{1}{(n-1)!}\lim_{z\to 0}\partial^n\hat{X}(z)|0\rangle \tag{2.13}$$

We can invert (2.11) using Cauchy theorem to find another form of the operator-state map

$$\hat{\alpha}_{-n}|0\rangle = i\sqrt{\frac{2}{\hat{\alpha}'}}\oint\frac{dz}{2\pi i}z^{-n}\partial\hat{X}(z)|0\rangle \tag{2.14}$$

*2.1.0.10 Momentum states*

Remember that $\alpha_0 \sim \hat{p}$ is the momentum quantum number. Then $\alpha_0|0\rangle = 0$ is nothing but the statement that we have chosen $|0\rangle$ to be the vacuum at zero momentum. A different vacuum with momentum $p$ can be obtained by means of the primary field $: e^{ipX(z)} :$ via the aforementioned operator-state map. For that define

$$|p\rangle = \lim_{z\to 0} : e^{ip\hat{X}(z)} : |0\rangle \tag{2.15}$$

Then let us check

$$\begin{aligned}
\hat{p}|p\rangle &= \sqrt{\frac{2}{\alpha'}}\hat{\alpha}_0 \left(\lim_{\omega\to 0} : e^{ip\hat{X}(\omega)} : |0\rangle\right) \\
&= \lim_{\omega\to 0}\sqrt{\frac{2}{\alpha'}}\left(i\sqrt{\frac{2}{\alpha'}}\right)\oint \frac{dz}{2\pi i}\,\partial\hat{X}(z) : e^{ip\hat{X}(\omega)} : |0\rangle \\
&= i\lim_{\omega\to 0}\frac{2}{\alpha'}\oint\frac{dz}{2\pi i}\left(\frac{-i\alpha' p}{2}\frac{: e^{ip\hat{X}(\omega)} :}{z-\omega} + ...\right)|0\rangle \\
&= p\lim_{\omega\to 0} : e^{ip\hat{X}(\omega)} : |0\rangle \\
&= p\,|p\rangle
\end{aligned} \tag{2.16}$$

## 2.2 $S$ matrix and Scattering Amplitudes

### 2.2.1 $m$-point amplitudes for closed strings.

*2.2.1.1 Vertex operators*

Given the form (2.3) of the string path integral, it is natural to generalize it to the computation of correlators in the form

$$\mathcal{A}^{(m)}(\Lambda_i, p_i) = \sum_{\text{topologies}} g_s^{-\chi}\int\frac{\mathcal{D}g}{Vol}\int(\mathcal{D}X)e^{-S_{\text{Polyakov}}}\prod_{i=1}^{m}\mathcal{V}(\Lambda_i, p_i) \tag{2.17}$$

where $\mathcal{V}(\Lambda_i, p_i)$ are the operators that insert the relevant asymptotic states at the punctures of a Riemann surface.

These are created by local operators called "vertex operators". They must be such that

- create states $|\Lambda, p\rangle$ of well defined momentum, hence that they transform with phase $e^{ip_i a}$ under shifts $X^\mu \to X^\mu + a^\mu$.
- have the correct Lorentz group tensorial structure.
- do not depend on the local coordinates chosen on the worldsheet
- be quantum-conformally invariant.

The guess is that $V_\Lambda(p)$ must be of the form

$$\hat{\mathcal{V}}(\Lambda,p)=\int d^2z\hat{V}(z,\bar{z};\Lambda,p) \tag{2.18}$$

where

$$\hat{V}(z,\bar{z};\Lambda,p)=:\Lambda(\hat{X}(z,\bar{z}))e^{ip\cdot\hat{X}(z,\bar{z})}: \tag{2.19}$$

and $\Lambda(\hat{X})$ is some differential polynomial in the embedding coordinates $X(z,\bar{z})$. For example, for a graviton with polarization $\zeta_{\mu\nu}=\zeta_{\nu\mu}$ it would be

$$\hat{\mathcal{V}}(\Lambda,p)=\int d^2z\,\zeta_{\mu\nu}\partial\hat{X}^\mu\bar{\partial}\hat{X}^\nu e^{ip\cdot\hat{X}}: \tag{2.20}$$

so we map *correlation functions* of a 2D CFT into *S-matrix elements* of a string theory.

*2.2.1.2 Mass spectrum from conformal invariance*

The residual conformal symmetry of the action (2.8) comes out because $dzd\bar{z}\partial_z\partial_{\bar{z}}$ is invariant under analytic transformations. In the same vein, for $\int d^2zV(z,\bar{z})$ to be invariant, the operator $V(z,\bar{z})$ has to transform as a $(1,1)$ complex tensor of weight $(h,\bar{h})=(1,1)$.

The (quantum) conformal weight of a vertex operator such as

$$\partial^N\bar{\partial}^{\bar{N}}\hat{X}(z,\bar{z})e^{ip\hat{X}(z,\bar{z})}$$

is $(N+\alpha'p^2/4,\bar{N}+\alpha'p^2/4)$ with $N=\bar{N}$ (level matching). From here we see that the condition of physical state puts the external momenta on the mass shell

$$M^2=-p^2=\frac{4}{\alpha'}(N-1) \tag{2.21}$$

The Regge behaviour stems from (2.21) together with the fact that, at level $N$ the highest spin that can be achieved comes from the fully symmetric tensor $\alpha_{-1}^{\mu_1}....\alpha_{-1}^{\mu_N}\bar{\alpha}_{-1}^{\nu_1}....\bar{\alpha}_{-1}^{\nu_N}|0\rangle$, hence

$$J_{\max}=2N=2+\frac{\alpha'}{2}M^2\,. \tag{2.22}$$

*2.2.1.3 Moduli space*

The integration over *moduli space* depends on the string diagram. Its dimension can be given in terms of the genus $h$, the number of boundaries $b$, and the amount of closed and open string insertions $n_c$ and $n_o$:

$$\mathcal{N}=3(2h+b-2)+2n_c+n_o\,. \tag{2.23}$$

This is also the subset of integrations $\prod_{i=1}^{\mathcal{N}}\int d^2z_i$ that must be skipped. For example, a 4 point function on a sphere base $\mathcal{N}=2$.

*2.2.1.4 Tree level amplitude*

At tree level we would write then

$$\begin{aligned}\mathcal{A}^{(m)}(\Lambda_i, p_i) &= \frac{1}{g_s^2}\frac{1}{\mathrm{Vol(SL(2,C))}}\int (\mathcal{D}X) e^{-\frac{1}{2\pi\alpha'}\int d^2z\ \partial X\cdot\bar{\partial}X}\prod_{i=1}^{m}\int d^2z_i V_i(z_i,\bar{z}_i) \\ &= \frac{1}{g_s^2}\frac{\prod_{i=1}^{m}\int d^2z_i}{\mathrm{Vol(SL(2,C))}}\ \langle V_1(z_1,\bar{z}_1)....V_1(z_m,\bar{z}_m)\rangle_{\mathrm{CFT}}\end{aligned} \tag{2.24}$$

Hence, essentially we are entitled to compute correlation functions of a scalar field on the sphere. The residual gauge symmetry is given by the global analytic transformations of the complex plane which are regular when $z \to \infty$. One can show that the group of such transformation is indeed $SL(2,C)$. This means global analytical transformations act on the coordinates $z_i$ and therefore relate many portions of the multidimensional integral. Dividing by the volume of this redundancy is tantamount to fixing the residual gauge symmetry.

Actually the $SL(2,C)$ has 3 independent complex parameters. This means we can fix the residual gauge symmetry by placing 3 punctures to desired positions. Let these be

$$z_1 = 0 \quad ; \quad z_2 = 1 \quad ; \quad z_4 = \infty\,. \tag{2.25}$$

and $z_3 = z$ is the true relevant *moduli* which parametrizes inequivalent contributions.

2.2.1.5 *SL(2, C) invariance*

*Lemma:* The only globally defined conformal transformations of the Riemann sphere are given, infinitesimally by

$$z \to z + \epsilon(\alpha + \beta z + \gamma z^2) + \mathcal{O}(\epsilon^2) \tag{2.26}$$

and globally as

$$z \to \frac{Az + B}{Cz + D} \qquad AB - CD = 1 \tag{2.27}$$

Proof: The infinitesimal action above is performed by the action of a vector field $z \to z + \epsilon V[z]$ with $V = V^z\partial_z = (a_i z^i)\partial_z$ Regularity at $z = 0$ demands $i \geq 0$. Regularity at $z = \infty$ must be investigaded by inverting $z \to z' = 1/z$ and demanding $V = V^{z'}\partial_{z'}$ to be regular at $z' = 0$. But

$$V^{z'}(z') = \frac{dz'}{dz}V^z(z(z')) = -z^{-2}a_i z'^{-i} = -a_i z'^{2-i} \tag{2.28}$$

which is regular at $z' = 0$ for $i \leq 2$.

From here, (2.26) is recovered doing $A = 1 + a\epsilon, B = b\epsilon, C = c\epsilon, D = 1 + d\epsilon$ and expanding to first order in $\epsilon$ it turns out that $\alpha = b$, $\beta = a - d$ and $\gamma = -c$. . ■

Hence, we can cast all the information of a global conformal transformation in terms of a complex matrix $\Delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of unit determinant $\det(\Delta) = 1$. Moreover, it is trivial to verify, that the composition of two conformal transformations is isomorphic to the multiplication of matrices. Hence the residual conformal symmetry is isomorphic to the Lie group $SL(2,C)$.

Notice that by the following change

$$z' = \frac{(z-z_1)(z_2-z_4)}{(z-z_4)(z_2-z_1)} \tag{2.29}$$

we may bring any 3 points $z_1, z_2$ and $z_4$ to values $0, 1$ and $\infty$ respectively.

### 2.2.2 4-point tachyon amplitude

In the bosonic string the simplest vertex operator is the one for the tachyon state $N = 0$ hence $M^2 = -4/\alpha'$.

$$\mathcal{V}(0\,; p) = g_s \int d^2z e^{ip\cdot X} = g_s \int d^2z\, V(z, \bar{z}; p) \tag{2.30}$$

where we have added a normalization factor $g_s$. Theory is free, hence Wick's theorem is all we have to do. Let us define the notation

$$X\cdot K\cdot X = \int d^2z d^2\omega\, X(z,\bar{z})K(z,\bar{z};\omega,\bar{\omega})X(\omega,\bar{\omega}) \quad ; \quad X\cdot J = \int d^2z\, J(z,\bar{z})X(z,\bar{z}) \tag{2.31}$$

then with $K = -\frac{1}{\pi\alpha'}\partial\bar{\partial}\delta^2(z-\omega)$, and $J^\mu = i\sum_j p_j^\mu \delta(z-z_i)$ we can write

$$\begin{aligned}
\langle\prod_{i=1}^m V(z_j,p_j)\rangle_{\text{CFT}} &= \langle \exp(J\cdot X)\rangle \\
&= \int(\mathcal{D}X)\exp\left[-\frac{1}{2}X\cdot K\cdot X + J\cdot X\right] \\
&= \int(\mathcal{D}\tilde{X})\exp\left[-\frac{1}{2}\tilde{X}\cdot K\cdot\tilde{X} + J\cdot\tilde{X}\right]\int(\prod_{\mu=0}^D dx_0^\mu)\exp\left[J\cdot x_0\right]
\end{aligned}$$

where we have separated $X^\mu(z,\bar{z}) = x_0^\mu + \tilde{X}^\mu(z,\bar{z})$. Performing now the shift $\tilde{X} = \tilde{Y} + G\cdot\tilde{X}$ where $G$ is defined by

$$\int d^2u\ G(z,u)K(u,\omega) = \delta^d(z-\omega) \tag{2.32}$$

$$\begin{aligned}
\langle\exp\left[J\cdot X\right]\rangle &= \int(\mathcal{D}\tilde{Y})\exp\left[-\frac{1}{2}\tilde{Y}\cdot K\cdot\tilde{Y} + \frac{1}{2}J\cdot G\cdot J\right]\int(\prod_{\mu=0}^D dx_0^\mu)\exp\left[J\cdot x_0\right] \\
&= Z[0]\exp\left[\frac{1}{2}J\cdot G\cdot J\right]\delta\left(\sum_i p_i\right) \\
&= \delta\left(\sum_i p_i\right)\prod_{j=1}^m d^2z_i\prod_{i,j}\exp\left[\frac{1}{2}p_iG(z_i,z_j)p_j\right]
\end{aligned} \tag{2.33}$$

The Green's function is given by

$$G(z,\bar{z};\omega,\bar{\omega}) = -\frac{\alpha'}{2}\ln|z-\omega|^2 \tag{2.34}$$

and so we end up with

$$\mathcal{A}^{(m)}(\Lambda_i,p_i) \;=\; \delta\left(\sum_i p_i\right)\frac{g_s^{m-2}}{\text{Vol}(SL(2,C))}\int\prod_{i=1}^{m}d^2z_i\prod_{j<l}|z_j-z_l|^{\alpha' p_j\cdot p_l} \tag{2.35}$$

Using the OPE's we can also arrive at the same result

*Theorem*

$$:e^{iaX(z)}::e^{ibX(\omega)}:=(z-\omega)^{ab}:e^{i(aX(z)+bX(\omega))}: \tag{2.36}$$

Proof:

The following lemma is just combinatorics, for any pair of operators

$$:A^m::B^n:=\sum_{l=0}^{\min(m,n)}\frac{m!n!}{(m-l)!l!(n-l)!}(\underline{AB})^l:A^{m-l}B^{n-l}: \tag{2.37}$$

Now

$$\begin{aligned}
:e^{iaX(z)}::e^{ibX(\omega)}: &= \sum_{m=0}^{\infty}\frac{:(iaX(z))^m:}{m!}\sum_{n=0}^{\infty}\frac{:(ibX(\omega))^n:}{n!}\\
&= \sum_{p=0}^{\infty}\frac{(-ab\,\underline{X(z)X(\omega)})^p}{p!}:\sum_{m,n=p}^{\infty}\frac{(iaX(z))^{m-p}}{(m-p)!}\frac{(ibX(\omega))^{n-p}}{(n-p)!}:\\
&= e^{\alpha' ab/2\,\log(z-\omega)}:e^{iaX(z)+ibX(\omega)}:\\
&= (z-\omega)^{\alpha' ab/2}:e^{iaX(z)+ibX(\omega)}:
\end{aligned} \tag{2.38}$$

■

With this we see that, for example

$$\begin{aligned}
\langle :e^{ip_1X(z)}::e^{ip_2X(\omega)}:e^{ip_2X(u)}:\rangle &= (z-\omega)^{p_2p_3}\langle :e^{ip_1X(z)}::e^{i(p_2X(\omega)+p_3X(u))}:\rangle\\
&= (z-\omega)^{p_2p_3}(z-\omega)^{p_1(p_2+p_3)}\langle :e^{i(p_1X(z)+p_2X(\omega)+p_3X(u))}:\rangle\\
&= (z-\omega)^{p_2p_3}(z-\omega)^{p_1(p_2+p_3)}\langle :e^{i(p_1+p_2+p_3)X(z)}:\rangle+....\\
&= (z-\omega)^{(p_2p_3+p_1p_2+p_1p_3)}\delta(p_1+p_2+p_3).
\end{aligned} \tag{2.39}$$

(in the above we have taken $\alpha'=2$ for convenience and normalized $\langle 0|0\rangle=1$.)

*2.2.2.1 4-point amplitude*

Setting $m=4$ we end up with [1]

$$\mathcal{A}^{(4)}(\Lambda_i,p_i) \;=\; \delta\left(\sum_i p_i\right)\frac{g_s^{2}}{\text{Vol}(SL(2,C))}\int\prod_{i=1}^{4}d^2z_i\prod_{j<l}|z_j-z_l|^{\alpha' p_j\cdot p_l} \tag{2.42}$$

[1]there is a factor

$$|0-z_4|^{\alpha' p_1p_4}|1-z_4|^{\alpha' p_2p_4}|z-z_4|^{\alpha' p_3p_4}\overset{z_4\to\infty}{\longrightarrow}|z_4|^{-\alpha' p_4^2}=|z_4|^{-4} \tag{2.40}$$

After fixing the $SL(2,C)$ invariance by putting the insertion points at $0, 1, z$ and $z_4 \to \infty$ we end up with

$$\mathcal{A}^{(4)} \sim g_s^2 \delta\left(\sum_i p_i\right) \int d^2z |z|^{\alpha' p_1 \cdot p_3} |1-z|^{\alpha' p_2 \cdot p_3} \tag{2.43}$$

using Gamma function identities this expression can be given a nice form. One must use the integral representation

$$\int d^2z |z|^{2a-2} |1-z|^{2b-2} = \frac{2\pi \Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} \tag{2.44}$$

where $a+b+c=1$. With this, (2.43) can be shown to be equal to

$$\mathcal{A}^{(4)} \sim g_s^2 \delta\left(\sum_i p_i\right) \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)} \tag{2.45}$$

in terms of the Mandelstam variables

$$s = -(p_1+p_2)^2 \quad ; \quad t = -(p_2+p_3)^2 \quad ; \quad u = -(p_1+p_4)^2 \tag{2.46}$$

which satisfy on shell (i.e. use the tachyon mass $-p_i^2 = M^2 = -4/\alpha'$)

$$s+t+u = -\sum_{i=1}^{4} p_i^2 = \sum M_i^2 = -\frac{16}{\alpha'} \tag{2.47}$$

<u>2.2.2.2</u> *Regge behaviour*

Consider the large $s$, fixed $t$ limit. The amplitude $\mathcal{A}^{(4)}$ has infinite poles in the $s$ channel at positions $s = 4(N-1)/\alpha'$. Hence at values given by states in the closed string mass spectrum, starting from the tachyon itself ($n=0$). Moreover, the residues of these poles have a leading behaviour (in $p$) that goes with $t^{2n}$ hence $\sim p^{4n}$.

Therefore, the *single* string amplitude is equivalent to a sum over field theory amplitudes where all the spectrum contributes, and at level $n$ with particles of maximum spin $J=2n$

$$\mathcal{A}^{(4)} \sim \sum_{n=0}^{\infty} \frac{t^{2n}}{s - M_n^2} + \mathcal{O}(t^{2n-1}) \tag{2.48}$$

We could also fix $s$ and let $t$ grow. We then find poles in the $t$ channel, corresponding to the same resonances in the spectrum. In field theory these two channels should be added up. Not so in string theory.

<u>2.2.2.3</u> *Soft high energy behaviour*

for a tachyonic mass $M^2 = -4/\alpha'$. To correctly cancel this factor one has to compute the Jacobian for the change of parameters $(z_1, z_2, z_4) \to (\alpha, \beta, \gamma)$ in (2.26). This is

$$\left|\frac{\partial(z_1, z_2, z_4)}{\partial(\alpha, \beta, \gamma)}\right|^2 = |(z_1-z_2)(z_1-z_4)(z_2-z_4)|^2 \xrightarrow{z_4 \to \infty} |z_4|^4 . \tag{2.41}$$

Sending $s, t \to \infty$ while keeping the scattering angle fixed we explore the short distance behaviour of the theory. In this limit one can show that

$$\mathcal{A}^{(4)} \sim g_s^2 \delta\left(\sum_i p_i\right) \exp\left(-\frac{\alpha'}{2}(s\log s + t\log t + u\log u)\right) \tag{2.49}$$

which does not diverge. This is in contrast with any truncation of (2.48) to a finite number of terms, which leads to a divergente behaviour.

- Mathematically, infinite sums can behave unlike finite sums.

- Physically, the string length $l_s = \sqrt{\alpha'}$ prevents the string from proving distances shorter than $l_s$.

*2.2.2.4 Graviton scattering*

One can compute also amplitudes for other states in the spectrum. For example, the graviton vertex operator

$$V(k) = \int d^2z\, \zeta^{\mu\nu}\partial_\mu X \partial_\nu X e^{ik\cdot X} \tag{2.50}$$

can be also used to compute tree level scattering in string theory. The result can be compared with the tree level graviton scattering coming from Einstein's action

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-G}\,\mathcal{R}\,. \tag{2.51}$$

The upshot is

- At high energies both answers differ. String theory amplitude converges while field theory diverges. This divergence is consistent with the dimensionality of the Newton's coupling constant $[G_{26}] = [\kappa^2] = l^{24}$.
- At energies $<< 1/\sqrt{\alpha'}$ both answers agree provided we identify

$$\kappa^2 = g_s^2(\alpha')^{12} \tag{2.52}$$

## 2.3 Open string scattering

The 2 dimensional Hilbert-Einstein action does not lead to well defined equations of motion when boundaries are present. The Gibbons-Hawking term must be added

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\sqrt{g}\,R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} ds\,\mathcal{K} \tag{2.53}$$

where $\mathcal{K}$ is the extrinsic curvature of the boundary

$$\mathcal{K} = -t^\alpha n_\beta \nabla_\alpha t^\beta \tag{2.54}$$

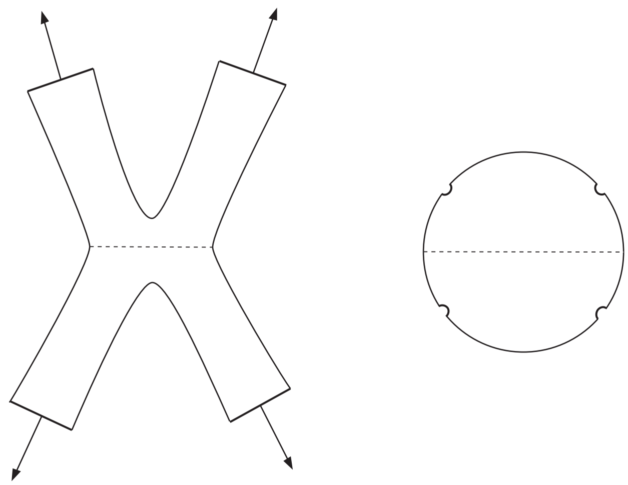


Figure 2.3: default

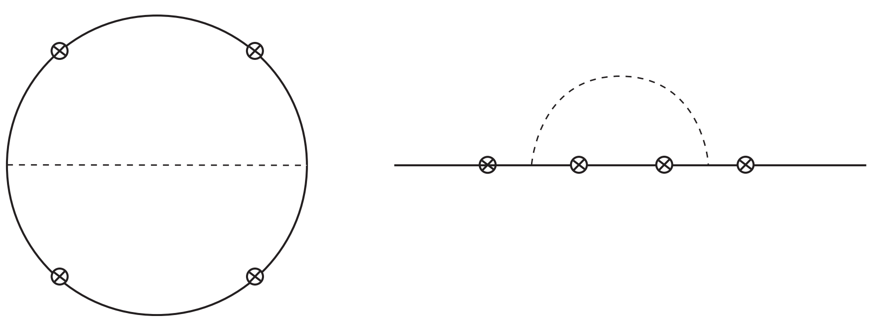


Figure 2.4: default

where $t^\alpha(n^\beta)$ is tangental (normal) to the boundary. Again $\chi$ is an integer (the Euler characteristic) for 2-d manifolds with boundary

$$\chi = 2 - 2h - b \tag{2.55}$$

with $h$ the number of handels (or genus $g$), and $b$ the number of boundaries. Hence $g_s^\chi$ weights the disc with a factor $g_s^{-1}$, the anulus with $g_s^0$ and so on.

In the same spirit as for the closed string, the role of infinite long tube there is here played by an infinitely long strip where open strings are attached at both past and future ends, that is $\omega = \sigma + i\tau$ with

$$-\infty \leq \tau \leq \infty \quad , \qquad 0 \leq \sigma \leq \pi \, , \tag{2.56}$$

where $\mathrm{Re}\,\omega = 0, \pi$ are the endpoint boundaries. Under $z = e^{-i\omega}$ this maps into the upper half plane, $\mathrm{Im}z \geq 0$, with an operator insertion at $z = 0$.

Now performing a conformal tranformation we may bring all the asymptotic insertions to finite distances of each other along the real axis, as we did with the closed string amplitude. Thereby we map the planar tree level diagram onto a disk with operators inserted at the boundary. To make the amplitude diffeomorphism invariant, the position on the boundary must be integrated over along the full real line.

$$\mathcal{V}(\Lambda, p) = \sqrt{g_s} \int_{-\infty+i0}^{+\infty+i0} dz\, \Lambda(X(z)) e^{ipX(z)} \tag{2.57}$$

were we have normalised the open string vertex by $g_{open} = \sqrt{g_{closed}} = \sqrt{g_s}$. The same calculation as with the closed string vertex operators is now in order. However the main difference is that $X(z, \bar{z})$ is only defined in the upper half plane. Suppose we want to implement Newmann boundary conditions. The propagator si defined as

$$\langle X(z, \bar{z}) X(z', \bar{z}') \rangle = G(z, \bar{z}; z', \bar{z}') \tag{2.58}$$

which obeys

$$\Box G = -2\pi\alpha' \delta(z - z', \bar{z} - \bar{z}') \, . \tag{2.59}$$

must be subject to the Newmann boundary condition

$$\partial_\sigma G(z, \bar{z}; z', \bar{z}')\big|_{\sigma=0,\pi} = 0 \, . \tag{2.60}$$

But notice that

$$\partial_\sigma \big|_{\sigma=0,\pi} = (-iz\partial_z + i\bar{z}\partial_{\bar{z}})|_{\mathrm{Im}\, z=0} = -\, i\mathrm{Re}\, z(\partial_z - \partial_{\bar{z}})|_{\mathrm{Im}\, z=0} \tag{2.61}$$

Hence Newman boundary conditions are the same as

$$\partial X(z) = \bar{\partial} \bar{X}(\bar{z}) \tag{2.62}$$

at $\mathrm{Im}z = 0$, and hence the same for $G(z, \bar{z})$. Hence, there are fewer states propagating along the world sheet, as the operators $\partial X$ and $\bar{\partial} X$ give rise to the same state (in contrast with the closed string where they were independent). Like in electrodynamics, we let $X(z,\bar{z})$ vary over the whole plane and add a "image charge" in the lower half plane. In other words, we let $X$ be defined on the whole complex plane, with dynamics given by the propagator

$$G(z,\bar{z};z',\bar{z}') = -\frac{\alpha'}{2}\log|z-z'|^2 - \frac{\alpha'}{2}\log|z-\bar{z}'|^2 \tag{2.63}$$

so, in complex coordinates $z = e^{\tau - i\sigma}$,

$$\begin{aligned}
\partial_\sigma G(z,\bar{z};z',\bar{z}') &= (-iz\partial_z + i\bar{z}\partial_{\bar{z}})G(z,\bar{z};z',\bar{z}') \\
&= i\frac{\alpha'}{2}\left(\frac{z}{z-z'} + \frac{z}{z-\bar{z}'} - \frac{\bar{z}}{\bar{z}-\bar{z}'} - \frac{\bar{z}}{\bar{z}-z'}\right) \\
&= i\frac{\alpha'}{2}\left(\frac{(\bar{z}-z)z'}{(z-z')(\bar{z}-z')} + \frac{(\bar{z}-z)\bar{z}'}{(z-\bar{z}')(\bar{z}-\bar{z}')}\right) \\
&= i\frac{\alpha'}{2}\left(\frac{z'}{(z-z')(\bar{z}-z')} + \frac{\bar{z}'}{(z-\bar{z}')(\bar{z}-\bar{z}')}\right)\mathrm{Im}(z)
\end{aligned} \tag{2.64}$$

which vanishes for $\mathrm{Im}(z) = 0$.

The amplitude is now computed with operator insertions along the boundary of the disk which maps onto the real axis of the complex plane. Hence there is a fixed ordering implicit in the amplitude. The Wick contraction part is the same with the modification of a factor 2 coming from the doubling in (2.63) caused by the image charge an there is a fixed order of operators which is kept invariant by the residual $SL(2,R)$ conformal Killing transformations that map the real line onto itself

$$\begin{aligned}
\mathcal{A}^{(4)} &= \frac{g_s}{\mathrm{Vol}(SL(2,R)}\int\prod_{i=1}^{4}dx_i\langle e^{ip_1\hat{X}(x_1)}\cdots e^{ip_1\hat{X}(x_4)}\rangle \\
&\sim \frac{g_s}{\mathrm{Vol}(SL(2,R)}\delta^{26}(\delta\sum_i p_i)\int\prod_{i=1}^{4}dx_i\prod_{j\leq l}|x_i - x_j|^{2\alpha' p_i\cdot p_j}
\end{aligned} \tag{2.65}$$

For a given ordering, the residual symmetry can be used to fix 3 points to $x_1 = 0$, $x_2 = 0$, $x_3 = x$ and $x_4 = \infty$. The resulting expression contains a single integration for $0 \leq x \leq 1$

$$\mathcal{A}^{(4)} \sim g_s\int_0^1 dx|x|^{2\alpha' p_1\cdot p_2}|1-x|^{2\alpha' p_2\cdot p_3}\,. \tag{2.66}$$

This integral is related to the Euler beta function

$$B(a,b) = \int_0^1 dx\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,. \tag{2.67}$$

Whence, using now the tachyon mass $M^2 = -1/\alpha'$ one recovers the *Veneziano amplitude*

$$\mathcal{A}^{(4)} \sim g_s\left(\frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-1-\alpha'(s+t))})\right) \tag{2.68}$$

The other channels come from considering all possible orderings of the vertex operators along the real line, whence the full symmetry $s \leftrightarrow t \leftrightarrow u$ is recovered.

$$\mathcal{A}^{(4)} \sim g_s \left[B(-1-\alpha' s, -1-\alpha' t) + B(-1-\alpha' s, -1-\alpha' u) + B(-1-\alpha' t, -1-\alpha' u)\right] \tag{2.69}$$

From the pole analysis of this formula $\alpha(s) = \alpha_0 + \alpha' s = n$ with an intercept $\alpha_0 = 1$ the resonances occur at

$$s = M^2 = \frac{N-1}{\alpha'} \tag{2.70}$$

which is precisely the spectrum of the open string. The level $N$ states now have highest angular momentum in the maximally symmetrized situation $\alpha_{-1}^{\mu_1} \cdots \alpha_{-1}^{\mu_N}|0\rangle$, whence $J = N$. The Regge behaviour of the open string is therefore of the form

$$J = 1 + \alpha' M^2 \tag{2.71}$$

# Chapter 3

# Gauge theory from string theory

## 3.1 Gauge theory from open strings

End points are the only non-ambiguous positions along open strings, always having $\sigma = 0, \pi$. We can assign to these disinguished points some new $N$ dimensional degree of freedom called *Chan-Paton factor*.

Each independent direction in this $N$ dimensional vector space is labelled $|i\rangle,\ i = 1, ..., N$.

A string state will be labelled in the form $|\Lambda, p\,; i, j\rangle = |\Lambda,\ p\rangle \otimes |i\rangle \otimes |j\rangle$ where $i, j$ are the charges placed at the endpoint. Therefore we will find $N^2$ possibilities and the whole spectrum will $N^2$-plicate

The worldsheet dynamics of these degrees of freedom is trivial, and they are conserved along the evolution of the free string. However they will have profound consequences for the spacetime physics.

In particular, the massles state $\alpha^{\mu}_{-1}|0, p\,;\ i, j\rangle$ will give a collection of $N^2$ massless vectors.

When such generalized strings interact they join and split their ends. It is natural to demand that after splitting, the two ends that split are in the same Chan Paton state

$$|\Lambda, p\,; i, j\rangle \to \sum_{k=1}^{N} c^{ijk} |\Lambda, p\,; i, k\rangle \oplus |\Lambda, p\,; k, j\rangle \tag{3.1}$$

An algebraic way to implement this is to define an alternative basis for the states

$$|\Lambda\,, p\,; a\rangle = \sum_{ij} \lambda^a_{ij} |\Lambda, p\,; i, j\rangle, \tag{3.2}$$

where $\lambda^a_{ij}$ is a basis of $N^2$ hermitian matrices subject to the condition $\mathrm{Tr}(\lambda^a \lambda^b) = \delta^{ab}$.

By the operator-state correspondence, all string vertex operators also carry such factor

$$\mathcal{V}(\Lambda, p, a)_{ij} = \int_{-\infty+i0}^{+\infty+i0} dz\, \lambda^a_{ij} : \partial X(z) e^{ipX(z)} : \tag{3.3}$$

Now an amplitude will have a sum over nearest neighboring coinciding CP factors, and this amounts to a trace over all matrices around the boundary in a given sense

$$\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}...\lambda^{a_3}) \tag{3.4}$$

All open string amplitudes will have now a trace factor like the one above, and will be invariant under a global (on the worldsheet) $U(N)$ under which the ends transform in the $N$ and $\bar{N}$ representation respectively.

$$\lambda^a \to U\lambda^a U^\dagger \qquad U^\dagger = U^{-1} \tag{3.5}$$

## 3.2 Gauge theory from closed strings

The massless sector of closed strings consists of gravitons, and axions, i.e. quanta of $G_{\mu\nu}$ and $B_{\mu\nu}$. Therefore we must somehow understand how gauge interactions can be obtained from gravity. In field theory this is achieved by compactifying dimensions. In string theory, compactification is a must, since we have too many dimensions to start with.

### 3.2.1 The Kaluza Klein reduction

Let us review the Kaluza Klein mechanism in its most elementary setup. Consider a 5 dimensional metric, and write it as the following fibration

$$ds^2 = G_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + e^{2\varphi}(dx^4 + A_\mu dx^\mu)^2 \tag{3.6}$$

that is

$$\begin{aligned} G_{\mu\nu} &= g_{\mu\nu} + e^{2\varphi}A_\mu A_\nu \qquad \mu,\nu = 0,...3, \\ G_{4\mu} &= e^{2\varphi}A_\mu \\ G_{44} &= e^{2\varphi} \end{aligned} \tag{3.7}$$

<u>3.2.1.1</u> *Gauge symmetry*

The original symmetry is just diffeomorphism invariance in 5 dimensions. Infinitesimally $\delta x^M = V^M(x)$ under which the metric transforms as $\delta G_{MN} = \nabla_M V_N + \nabla_N V_M$. Hence, a diffeomorphism in the compact 4'th direction $\delta x^4 = V^4(x^\mu) = \Lambda(x^\mu)$ turns into a gauge transformation of $A_\mu$

$$\delta A_\mu = \partial_\mu \Lambda(x^\mu) \tag{3.8}$$

<u>3.2.1.2</u> *Maxwell action*

Let us assume that $x^4$ *is periodic* $x^4 = x^4 + 2\pi R$ and *there is no dependence* on $x^4$. Then Einstein action in five dimensions can be rewritten as follows

$$\begin{aligned} S_5 &= \frac{1}{16\pi G^N_{(5)}}\int \sqrt{-G}\; R^{(5)} d^5x \\ &= \frac{1}{16\pi G^N_{(4)}}\int \sqrt{-G}\, e^{\varphi}\left(R^{(4)} + \partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{2\phi}F_{\mu\nu}F^{\mu\nu}\right) d^4x \end{aligned} \tag{3.9}$$

where the $x^4$ coordinate integrates to a trivial lenght factor $\int dx^4 = 2\pi R$, so that the effective Newton's constants are related by

$$\frac{1}{G^N_{(4)}} = \frac{2\pi R}{G^N_{(5)}} \tag{3.10}$$

*3.2.1.3 The spectrum*

If we do not neglect a possible dependence on the $x^4$ coordinate, any 5-dimensional scalar or field component which satisfies $\partial^M \partial_M \phi = 0$ can be expanded in Fourier harmonics

$$\phi(x^\mu) = \sum_{n \in Z} \phi_n(x^\mu) e^{inx^4/R} \quad . \tag{3.11}$$

Since $x^4$ is compact, the momentum in this direction is quantized $p_4 = n/R$, giving

$$\partial^\mu \partial_\mu \phi_n = \frac{n^2}{R^2} \phi_n \tag{3.12}$$

hence a tower of states with masses

$$M = \frac{n}{R} \tag{3.13}$$

so that the zero mode is massless. The mass splitting is controlled by the inverse radius $1/R$.

### 3.2.2 Closed strings on a circle

Let us consider a bosonic string propagating in a background $R^{1,25} \times S^1$. This amounts to identifying

$$X^{25} = X^{25} + 2\pi R \tag{3.14}$$

There are two ways in which this affects dynamics

- string momenta are quantized

  $$p^{25} = \frac{n}{R} \tag{3.15}$$

  this can be seen because the string wavefunction contains the factor $e^{ip \cdot X}$.

- as we move around the string we may wrap around the circle $m$ times

  $$X^{25}(\sigma + 2\pi) = X^{25} + 2\pi m R \, . \tag{3.16}$$

  $m$ is the *winding number*

We must change the zero mode expansion in order to incorporate these possibly generalized boundary conditions

$$X^{25}(\sigma, \tau) = x^{25} + \frac{\alpha' n}{R} + mR\sigma + \text{oscillator modes} \, . \tag{3.17}$$

Remember the most general solution to the equations of motion $\partial_+\partial_-\phi = 0$ is given by

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma-\tau) + X^\mu_R(\sigma+\tau)\,. \tag{3.18}$$

In order to split (3.17) into left and right moving modes, we will define

$$p_L^{25} = \frac{n}{R} + \frac{mR}{\alpha'} \quad ; \quad p_R^{25} = \frac{n}{R} - \frac{mR}{\alpha'} \tag{3.19}$$

where, for the closed string we have expansions $\sigma \in (0, 2\pi)$

$$\begin{aligned}
X_L^{25} &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p_L^{25}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^{25}}{n}e^{-in(\tau-\sigma)} \\
&\overset{\tau\to -i\tau}{=} \frac{1}{2}x^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^{25}(-i\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^{25}}{n}e^{-n(\tau-i\sigma)} \\
&= \frac{1}{2}x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^{25}\log z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^{25}}{n}z^{-n} \qquad (3.20) \\
X_R^{25} &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p_R^{25}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde\alpha_n^{25}}{n}e^{-in(\tau+\sigma)} \\
&\overset{\tau\to -i\tau}{=} \frac{1}{2}x^\mu + \sqrt{\frac{\alpha'}{2}}\tilde\alpha_0^{25}(-i\tau+\sigma) + i\sqrt{\frac{\tilde\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde\alpha_n^{25}}{n}e^{-n(\tau+i\sigma)} \\
&= \frac{1}{2}x^\mu - i\sqrt{\frac{\alpha'}{2}}\tilde\alpha_0^{25}\log \bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde\alpha_n^{25}}{n}\bar z^{-n} \qquad (3.21)
\end{aligned}$$

with

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}}\,p_L \quad ; \quad \tilde\alpha_0 = \sqrt{\frac{\alpha'}{2}}\,p_R\,. \tag{3.22}$$

<u>3.2.2.1</u> *Mass spectrum*

$$\begin{aligned}
L_0 &= \frac{1}{2}\alpha_0^\mu\alpha_{0\mu} + \frac{1}{2}(\alpha_0^{25})^2 + \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n \\
&= \frac{\alpha'}{4}p^\mu p_\mu + \frac{1}{2}(\alpha_0^{25})^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n - \frac{1}{24}24 \\
&= \frac{\alpha'}{4}p^\mu p_\mu + \frac{\alpha'}{4}\left(\frac{n}{R}+\frac{mR}{\alpha'}\right)^2 + N - 1 \qquad (3.23)
\end{aligned}$$

Hence $L_0|\text{phys}\rangle = \tilde L_0|\text{phys}\rangle = 0$ amounts to

$$\begin{aligned}
M^2 &= \left(\frac{n}{R}+\frac{mR}{\alpha'}\right)^2 + \frac{4}{\alpha'}(N-1) \qquad (3.24) \\
&= \left(\frac{n}{R}-\frac{mR}{\alpha'}\right)^2 + \frac{4}{\alpha'}(\tilde N-1) \qquad (3.25)
\end{aligned}$$

Adding and subtracting this can be equivalently stated as

$$H|\text{phys.}\rangle = (L_0 + \tilde{L}_0)|\text{phys.}\rangle \;=\; 0 \quad \Rightarrow M^2 = \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2} + \frac{4}{\alpha'}(N-1) \quad (3.26)$$

$$P|\text{phys.}\rangle = (L_0 - \tilde{L}_0)|\text{phys.}\rangle \;=\; 0 \quad \Rightarrow nm + N - \tilde{N} = 0 \quad (3.27)$$

*3.2.2.2 Field theory limit*

In the limit $\alpha' \to \infty$ what we find is that the winding modes become infinitely massive, and just the $m = 0$ mode survives (it costs infinite energy to wrap the string). The spectrum is the same as for the Kaluza-Klein states.

### 3.2.3 Massless spectrum

Setting $M^2 = 0$ in (3.24) and (3.24) is solved by

$$n = m = 0 \quad ; \quad N = \tilde{N} = 1 \quad (3.28)$$

hence an unwounded restless $(1,1)$ oscillator state. Given the index choices $M = (\mu, 25)$ we have the following possibilities for

$$\mathcal{V}(\Lambda; p) = \int d^2z\Lambda(z,\bar{z})e^{ip\cdot X(z,\bar{z})} \quad (3.29)$$

| field | state | $\Lambda(z,\bar{z})$ |
|---|---|---|
| $G_{\mu\nu}$ | $(\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1} + \alpha^\nu_{-1}\tilde{\alpha}^\mu_{-1})$ | $\partial X^\mu_L\bar{\partial}X^\nu_R + \partial X^\nu_L\bar{\partial}X^\mu_R$ |
| $B_{\mu\nu}$ | $(\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1} - \alpha^\nu_{-1}\tilde{\alpha}^\mu_{-1})$ | $\partial X^\mu_L\bar{\partial}X^\nu_R - \partial X^\nu_L\bar{\partial}X^\mu_R$ |
| $A_{\mu(L)}$ | $\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1}$ | $\partial X^\mu_L\bar{\partial}X^{25}_R$ |
| $A_{\nu(R)}$ | $\alpha^{25}_{-1}\tilde{\alpha}^\nu_{-1}$ | $\partial X^{25}_L\bar{\partial}X^\mu_R$ |
| $\phi = \frac{1}{2}\log G_{25,25}$ | $\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}$ | $\partial X^{25}_L\bar{\partial}X^{25}_R$ |

*Massless states and their vertex operators*

## 3.3 T- duality

A new, intrinsically stringy effect comes about. Notice the following symmetry of the compact case:

$$m \leftrightarrow n \quad ; \quad R \leftrightarrow \frac{\alpha'}{R} \quad ; \quad \alpha^{25}_n \leftrightarrow -\alpha^{25}_n \quad (3.30)$$

This can also be seen in the L-R basis, where it amounts to a chiral reflection

$$X^{25}_L \to X^{25}_L \quad ; \quad X^{25}_R \to -X^{25}_R \quad (3.31)$$

### 3.3.1 Full spectrum: T-duality for closed strings

When $R \to \infty$ the momentum states ($p = n/R, m = 0$) become light. On the contrary, when $R \to 0$ the winding states ($p = 0, w = mR$) become light. Under T-duality these states go over to momentum states of the dual theory ($p = m/R', w = 0$) with $R' = \alpha'/R \to \infty$. Hence the wave packets built out of light states always signal a periodicity larger than the self-dual radius $\sqrt{\alpha'}$.

### 3.3.2 Enhanced gauge symmetry:

At the selfdual radius $R = \sqrt{\alpha'} = R'$, inserting into (3.24) and (3.25) we find the massless spectrum given by solving

$$(n+m)^2 + 4N = 4 \quad ; \quad (n-m)^2 + 4\tilde{N} = 4 \tag{3.32}$$

In addition to the massless states

$$n = m = 0 \qquad ; \qquad N = \tilde{N} = 1$$

shown above, we find the following solutions with nontrivial momentum and winding $(p_L^{25}, p_R^{25}) = \pm 2/\sqrt{\alpha'}$

$$n = -m = \pm 1 \; ; \; N = 1 \; , \; \tilde{N} = 0 \quad ; \qquad n = m = \pm 1 \; ; \; N = 0 \; , \; \tilde{N} = 1 \tag{3.33}$$

Let us group the corresponding vertex operators are as follows

$$\mathrm{SU}(2)_L \qquad \left[ \partial X_L^\mu \, , \, \exp(\pm i \frac{2}{\sqrt{\alpha'}} X_L^{25}) \right] \bar{\partial} X_R^\mu \tag{3.34}$$

$$\mathrm{SU}(2)_R \qquad \partial X_L^\mu \left[ \bar{\partial} X_R^{25} \, , \, \exp\left( \pm i \frac{2}{\sqrt{\alpha'}} X_R^{25} \right) \right] \tag{3.35}$$

This is because we can only make sense of massless vectors as gauge fields. Moreover the momentum states are charged under the operator creating an oscillator state $\sim \partial X$

$$\begin{aligned}
[H, W_\pm] &= [\oint \frac{dz}{2\pi i} \partial X(z), \oint \frac{dw}{2\pi i} e^{\pm i \frac{2}{\sqrt{\alpha'}} X(\omega)}] = \oint_0 \frac{d\omega}{2\pi i} \oint_\omega \frac{dz}{2\pi i} \partial X(z) e^{\pm i \frac{2}{\sqrt{\alpha'}} X(\omega)} \\
&= \oint_0 \frac{d\omega}{2\pi i} \oint_\omega \frac{dz}{2\pi i} i \frac{\alpha'}{2} \frac{\frac{\pm 2}{\sqrt{\alpha'}}}{z - \omega} e^{\pm i p X(\omega)} \\
&= \pm i \sqrt{\alpha'} W_\pm
\end{aligned} \tag{3.36}$$

Therefore, the states $W_\pm$ have the correct charges to be interpreted as $W$ bosons of the $SU(2)$ gauge symmetry.

# Chapter 4

# D-branes

As we have seen in the previous section, $T$-duality maps symmetrically physics on radius $R$ to physics on radius $\tilde{R} = \sqrt{\alpha'}/R$. In this way, letting $R \to 0$ does not imply losing a coordinate, since the continuum of winding states becomes massless and allows to explore the dual geometry which is almost uncompactified $\tilde{R} \to \infty$. Hence closed strings are always 26 or 10 dimensional. Open strings however, do not hace winding modes, and hence when $R \to 0$ they effectively live in one dimension less. However the interior of an open string is made of the same stuff as that of closed strings, and hence still vibrates in 10 dimenions. The clue is the dimenisonal reduction effectively only affects to the boundary points.

## 4.1 T-duality for open strings and D-branes

Remember the mode expansion of the open string with Neumann boundary conditions

$$X^M(\tau,\sigma) = x_0^M + 2\alpha' p^M \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^M}{n} e^{-in\tau} \cos(n\sigma) \tag{4.1}$$

which satisfies

$$\partial_\sigma X^M(\tau, 0) = \partial_\sigma X^M(\tau,\pi) = 0\,. \tag{4.2}$$

This can be expressed as

$$X^M(\tau,\sigma) = X_L^M(\tau+\sigma) + X_R^M(\tau-\sigma) \tag{4.3}$$

where

$$X_{L,R}^M(\tau \pm \sigma) = \frac{x_0^M}{2} \pm \frac{x_0'^M}{2} + \alpha' p^M(\tau\pm\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n}{n} e^{-in(\tau\pm\sigma)} \tag{4.4}$$

Putting the string on a compact dimension as before quantizes the momentum

$$X^{25}(\tau,\sigma+\pi) \sim X^{25}(\tau,\sigma) + 2\pi R \quad \Rightarrow \quad p^{25} = \frac{n}{R} \tag{4.5}$$

as there are now winding states available for open strings.

Since the interior of an open string is indistinguishable from that of a closed string, we may play the same T-duality rule and define the dual coordinate

$$\begin{aligned} X'^{25}(\tau,\sigma) &= X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma) \\ &= X_0'^{25} + 2\alpha' \frac{n}{R}\sigma + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^M}{n}e^{-in\tau}\sin(n\sigma) \end{aligned} \tag{4.6}$$

This dual coordinate has fixed endpoints

$$\partial_\tau X'^{25}(\tau,0) = \partial_\tau X'^{25}(\tau,\pi) = 0\,. \tag{4.7}$$

We see that $T$-duality consistently maps Newman into Dirichlet boundary conditions. In fact we see that $X_0'^{25}$ is one of the string ends. From the mode expansion, the other end sits at the same point, possibly shifted by an integer multiple of the periodicity

$$\begin{aligned} X'^{25}(\tau,0) &= X_0'^{25} \\ X'^{25}(\tau,\pi) &= X_0'^{25} + \frac{2\pi\alpha' n}{R} \end{aligned} \tag{4.8}$$

Hence in the same vein as for closed strings, *momenta* go in the dual picture over to *winding*, and for this to be consistent the enpoints have to be fixed. The winding comes about because it is the interior of the string that feels the full coordinate and therefore can wrap around it, whereas the end points do not. In short, the $T$ dual of an open string propagating on a compact dimension is another string propagating on a D-brane.

### 4.1.1 D branes

We will interpret the original ten dimensional open strings as lying on a D25-brane that fills the (25+1) dimensional spacetime. If we compactify one coordinate, say $X^{25}$ and send $R \to 0$, we can intepret the end result through duality as living again in 25 dimensiones but *with enpoints confined to a co-dimension one hypervolume* hence to a D24 brane.

In general a T duality transformation along a direction parallel (perpendicular) to a D$p$-brane, produces a D$(p-1)$-brane (D$(p+1)$-brane).

### 4.1.2 T-duality with Wilson Lines

Consider adding $U(N)$ Chan-Paton factors. When we compacitfy $X^{25}$ we can turn on a constant *vev* for the gauge field $A_{25}(X^{25})$ as follows

$$A_{25} = \frac{1}{2\pi R}\mathrm{diag}\,(\theta_1,\theta_2,...\theta_N) \tag{4.9}$$

wich is a pure gauge field $A_{25} = -i\Lambda^{-1}\partial_{25}\Lambda$

$$\Lambda(X^{25}) = \mathrm{diag}\left(e^{i\theta_1 X^{25}/2\pi R}, e^{i\theta_2 X^{25}/2\pi R}, ..., e^{i\theta_N X^{25}/2\pi R}\right)\,. \tag{4.10}$$

In a state $|k;ij\rangle$ the end-points couple to the vector potential like a point particle. Hence upon going around the circle, we only require quasiperiodicity of states

$$|k;ij\rangle \longrightarrow e^{i(\theta_i-\theta_j)}|k;ij\rangle \tag{4.11}$$

For Newman boundary conditions, this quantizes momenta as follows

$$p^{25} = \frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R} \tag{4.12}$$

and hence, the dual Dirichlet boundary conditions are now (for an initial $b = 0$)

$$X'^{25}(\tau,\sigma) - X'^{25}(\tau,0) = (2\pi n + \theta_j - \theta_i)R' \tag{4.13}$$

Hence, T-duality maps Wilson lines, to positions of dual D-branes.

### 4.1.3 T-duality for type II superstrings

When going over to superstrings, we first have a new critical dimension $10 \to 25$. World sheet supersymmetry T-duality, which acts as a right handed parity transformation $X^9_R \to -X^9_R$ must act likewise on the fermion partners

$$T \ : \ \psi^9_- \to -\psi^9_- \tag{4.14}$$

Let us see how this acts on the GSO projection.

*4.1.3.1 Summary of Type IIA and Type IIB.*

Remember that fermions admit the following mode expansions

$$\begin{aligned} \psi_\pm(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_r \psi^\mu_r e^{-ir(\tau\pm\sigma)} \\ r &= 0, \pm 1, \pm 2, ...(\mathrm{R}) \\ &= \pm\frac{1}{2}, \pm\frac{3}{2}, .......(\mathrm{NS}) \end{aligned} \tag{4.15}$$

In order to get rid of the tachyon we project out half of the spectrum by means of the GSO projector.

- *NS sector:*

$$P_{NS} = \frac{1}{2}\left(1 - (-1)^F\right) \quad ; \quad F = \sum_{r=\frac{1}{2},\frac{3}{2},...} \psi^i_{-r}\psi^i_r \tag{4.16}$$

  This allows states with $F = 1, 3, 5..$ hence with $N = \sum_r r\psi^i_{-r}\psi^i_r = \frac{1}{2}, \frac{3}{2}, ...$ hence masses $m^2_{NS} = \frac{1}{\alpha'}(N - \frac{1}{2}) = \frac{0}{\alpha'}, \frac{1}{\alpha'}, \frac{2}{\alpha'}, ...$

- *R sector*

  In this sector we have zero modes $\psi^\mu_0 = \frac{1}{\sqrt{2}}\Gamma^\mu$ which act as finite dimensional Dirac matrices. Now we have two possible GSO projectors

$$P^\pm_R = \frac{1}{2}\left(1 \pm \Gamma^{11}\cdot(-1)^F\right) \quad ; \quad F = \sum_{r=1,2,...} \psi^i_{-r}\psi^i_r \tag{4.17}$$

where $\Gamma^{11}=\Gamma^0\Gamma^1...\Gamma^9$ is the 10-dimensional chirality operator $(\Gamma^{11})^2=\{\Gamma^\mu,\Gamma^{11}\}=0$.

$$\Gamma^{11}|\text{phys}\rangle_R=\pm|\text{phys}\rangle_R\,.$$

Depending on the sign choice, the surviving states have $F=0,2,4..$ and positive or negative chirality. The massless state is the vacuum, and has always $N=0$ and a well defined chirality. Massive states arise at levels $N=\sum_{r=1,2,...}r\psi^i_{-r}\psi^i_r=0,1,2,...$, hence $m^2_R=\frac{0}{\alpha'},\frac{1}{\alpha'},....$

For the open string the choice of $P^+_R$ or $P^-_R$ is a matter of taste. It yields two equivalent chiral theories with opposite spacetime chirality.

For the closed string this is also the case in the NS-NS, NS-R, and R-NS spectrum. However the R-R sector has now two inequivalent choices: same (Type IIB), or opposite (Type IIA) projections on both $left$ and $right$ handed sectors. In the RR sector the states correspond to low energy fields of the form $F^{(p+2)}=dC^{(p+1)}$. Let us see this.

The ground state of a Ramonf sector is a Majorana-Weyl spinor, which has $2^{10/2}/(2\times 2)=8$ real components. The two inequivalent representation of Spin(10) are usually termed $8_s$ and $8_c$. From a purely group theoretical perspective, the tensor product of these representations yields the following structure

$$\begin{aligned}\text{type IIA}\quad&\rightarrow\quad 8_s\otimes 8_c=[0]\oplus[2]\oplus[4]\\ \text{type IIB}\quad&\rightarrow\quad 8_s\otimes 8_s=[0]\oplus[2]\oplus[4].\end{aligned}\tag{4.18}$$

where $[p]$ denotes the antisymmetrized dimension $p$ irreducible representation of $Spin(10)$. This representation is in fact one in a basis of products of Dirac matrices with

$$\begin{aligned}\text{Type IIA}\quad&\rightarrow\quad|\psi^+_a\rangle\otimes|\psi^-_{\bar b}\rangle_R=\sum_{p=0,2,4}F^{(p)}_{\mu_1...\mu_p}(\Gamma^{[\mu_1}...\Gamma^{\mu_p]})_{a\bar b}\\ \text{Type IIB}\quad&\rightarrow\quad|\psi^+_a\rangle\otimes|\psi^-_b\rangle_R=\sum_{p=1,3,5}F^{(p)}_{\mu_1...\mu_p}(\Gamma^{[\mu_1}...\Gamma^{\mu_p]})_{ab}\end{aligned}\tag{4.19}$$

It is an exercise to show that the Dirac-Ramond equation

$$(G_0+\bar G_0)|\psi^+\rangle_R\otimes|\psi^\pm\rangle_R=0\tag{4.20}$$

implies the following equation for the field coefficients

$$\partial^\mu F^{(n)}_{\mu\mu_2...\mu_n}=0\quad;\quad\partial_{[\mu}F^{(n)}_{\mu_1...\mu_n]}=0\tag{4.21}$$

These equations are nothing but the statement that $F^{(n)}$ are field strengths for some $R-R$ Maxwell potentials $F^{(n)}\sim dC^{(n-1)}$.

$$\begin{aligned}\text{Type IIA}\quad&\rightarrow\quad C^{(1)},C^{(3)},C^{(5)},C^{(7)},C^{(9)}\\ \text{Type IIB}\quad&\rightarrow\quad C^{(0)},C^{(2)},C^{(4)},C^{(6)},C^{(8)}\end{aligned}\tag{4.22}$$

Actually, due to an identity of 10-dimensional Gamma matrices

$$\Gamma^{11}\Gamma^{[\mu_1 \dots \mu_n]} = \frac{(-1)^{[\frac{n}{2}]} n!}{(10-n)!} \epsilon^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_{10-n}} \Gamma^{[\nu_1 \dots \nu_{10-n}]} \tag{4.23}$$

the identity

$$\Gamma^{11} |\psi^{\pm}\rangle_R = \pm |\psi^{\pm}\rangle_R \tag{4.24}$$

amounts to a constraint

$$F^{(n)}_{\mu_1 \dots \mu_n} \sim \epsilon_{\mu_1 \dots \mu_n}{}^{\nu_1 \dots \nu_{10-n}} F^{(10-n)}_{\nu_1 \dots \nu_{10-n}} \tag{4.25}$$

which actually implies that $F^{(n)}$ and $F^{(10-n)}$ are not independent but related by *electric-magnetic* Hodge duality. In fact $F^{(5)}$ is self dual. This in turn implies that $C^{(p)}$ and $C^{(8-p)}$ are the electric and magnetic dual potentials.

The appearance of the potentials $C^{(p+1)}$ calls for the existence of some $p-$dimensional objects ($p-$branes) that couple minimally to them along some $p+1$ dimensional world-volume

$$q \int dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}} C^{(p+1)}_{\mu_1 \dots \mu_{p+1}} \,. \tag{4.26}$$

*4.1.3.2 T-duality*

From 4.14 we see that $T : \psi^9_{-,0} \to -\psi^9_{-,0}$, hence $T : \Gamma^{11} \to -\Gamma^{11}$ only on the right handed sector. Therefore, $T$ duality does not affect $P_{NS}$ but flips $T : P^{\pm}_R \to P^{\mp}_R$ only on the *right* handed sector, and so it will map Type IIB onto Type IIA strings and viceversa. Hence on one hand, it should map $C^{(p+1)} \to C^{(p)}$ potentials.

On the other hand, we have seen that T-duality along a D$p$-brane coordinate reduces this to a D$(p-1)$ brane, and viceversa. This is consistent with the fact that D$p$-branes couple to $C^{(p+1)}$ potentials.

D$p$ branes are the searched for objects that couple to the Ramond-Ramond potentials $C^{(p)}$. Therefore depending on the dimensionality they live in different Type II superstring theories

- D$(2p)$ branes exist in Type IIA
- D$(2p+1)$ branes exist in Type IIB

## 4.2 Dynamics of $D - branes$

D-branes are not rigid hyperplanes. They can be perturbed and the fluctuations are described by some interesting field theory of collective modes. Let us consider a $1+9$ dimensional string theory and T-dualize the coordinate $X^9$. A Chan Paton state $|k, ij\rangle$ is viewed from the T-dual perspective as a string stretching among D-branes positioned at values $X^9_i$, $i = 1, 2, \dots$. Denoting with $N$ the occupation number, the mass shell relation reads

$$\begin{aligned} m_{ij}^2 &= p_{ij}^2 + \frac{1}{\alpha'}(N-1) \\ &= \frac{L_{ij}^2}{(2\pi\alpha')^2} + \frac{1}{\alpha'}(N-1) \end{aligned} \tag{4.27}$$

where

$$L_{ij} = |2\pi n + (\theta_i - \theta_j)|R' \,. \tag{4.28}$$

As $R' \to \infty$ the contribution due to the string tension of strings stretching among different D-branes diverges. Hence only configurations with $i = j$ will lead to massless states which can be characterised as follows.

- $\alpha_{-1}^{\mu}|k, ii\rangle \leftrightarrow V = \partial_{\|} X^{\mu}$ .

  These states correspond to plane waves of gauge fields $A^{\mu}(\xi^a)$ where $\mu, a = 0, 1, ..., p$ and $\xi^a = x^a$ are coordinates on the D-brane world volume (static gauge). The fact that these gauge fields are confined to a domain wall in space-time given at some fixed value of $X^9$ lets us interpret D-branes as a kind go topological soliton or defect in space-time. The quanta of $A_{\mu}(\xi)$ describe fluctuations of internal degrees of freedom of such object.

- $\alpha_{-1}^{m}|k, ii\rangle \leftrightarrow V = \partial_{\|} X^{\mu} = \partial_{\perp} X'^{m}$

  These states correspond to scalar fields from the point of view of the SO(1,p) Lorentz group. $\Phi^m$, $m = p + 1, ..., 9$ originate from the gauge field of the parent string theory, describe the transverse position of the D-branes in the compact transverse dimensions. They describe fluctuations of the shape of the D-brane as embedded in space-time, in the same form as the original coordinates $X^I(\sigma, \tau)$ did for the string embedding shape.

### 4.2.1 Born-Infeld action

WE have seen the appearance of gauge fields living in the world volume of D-branes. ¿What sort of gauge theories do they describe? The physical picture of a D-brane fluctuation is the following: at certain position of the D-brane an open string pops up from the vacuum, evolves in time and later on disappears again. SInce the interior of the string is not forced to live on the D-brane, the world sheet will be that of a cigar-shaped worldsheet which is attached to the D-brane along a closed loop. Just the end points of the string are charged under the worldvolume gauge fields and, therefore, we will write for the bosonic string action in the conformal gauge

$$S[X, A] = \frac{1}{4\pi\alpha'} \in d^2 z \partial_z X^{\mu} \partial_{\bar{z}} X_{\mu} - i \int_0^{2\pi} d\theta \, \dot{X}^{\mu} A_{\mu} \,. \tag{4.29}$$

where $X^{\mu}(\theta)$ describes the shape of the boundary intersection and we have used the minimal coupling of a charged particle to a gauge field.

The task now is to perform a path integral over $X^\mu$, and obtain therefore a quantum effective action $S[A]$ for the gauge field

$$S[A] = \frac{1}{g_s} \int \mathcal{D}X e^{-S[X,A]} \tag{4.30}$$

A straightforward but tedious computation yields the following answer in $p = 9$

$$S[A] = \frac{1}{(4\pi^2\alpha')^5 g_s} \int d^{p+1}\xi \sqrt{\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}\,. \tag{4.31}$$

This result is exact in $\alpha'$ which is the coupling constant of the original two dimensional world sheet field theory. It is however tree level in the string coupling constant $g_s$. The coefficient in front of the action has the interpretation of a membrane tension. For general $p$ the result is

$$T_p = \frac{1}{(2\pi)^p \alpha'^{(p+1)/2} g_s} \tag{4.32}$$

The action (4.31) is known in the literature as the Born-Infeld action and its formulation dates back to 1934. These authors proposed it as a non-linear modification of the Maxwell electrodynamics which avoided the divergence of the energy of an electric field produced by a charged point like particle. Actually the maximum electric field in this theory is given by $1/(2\pi\alpha')$, whence we see again the softening properties of the finite string length. For small $\alpha' \to 0$ the BI action reduces to the Maxwell action.