String Theory

Lecture 1

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

Alfonso V. Ramallo

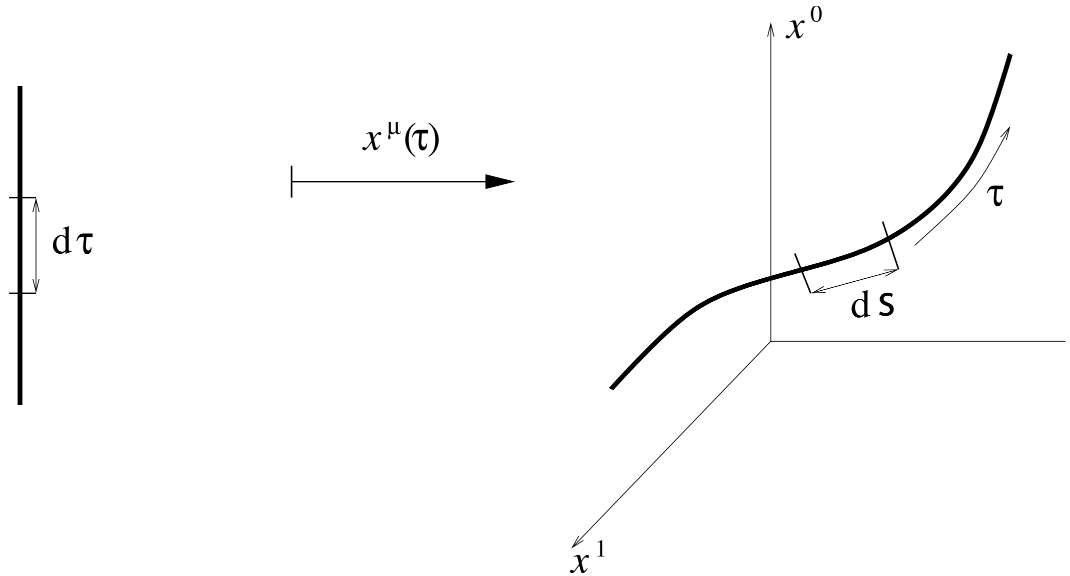
The relativistic point particle

$$\eta_{\mu\,\nu} \;=\; \mathrm{diag}\,(-1, +1, \cdots, +1)$$

$$x^{\mu} \;=\; x^{\mu}(\tau)$$

$x^{\mu} \rightarrow$ parametrizes the space in which the point particle is moving

$\tau \rightarrow$ coordinate that parametrizes the path of the particle



Action

$$\boxed{S = \int \mathcal{L}\, d\tau = -m \int ds = -m \int_{\tau_0}^{\tau_1} d\tau\, \sqrt{-\eta_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu}}$$

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} \qquad \eta_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu = \dot{x}^\mu\, \dot{x}_\mu \equiv \dot{x}^2$$

Lagrangian density

$$\boxed{\mathcal{L} = -m\sqrt{-\dot{x}^2}}$$

Conjugate momentum

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}$$

EOM

$$\boxed{\partial_\tau \left[\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right] = 0 \quad \Longrightarrow \quad \partial_\tau \left[\frac{m\,\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\right] = 0}$$

Constraint

$$p_\mu \, p^\mu \, + \, m^2 \, = \, 0$$

Hamiltonian analysis

$$H_{canonical} \, = \, \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \, \dot{x}^\mu \, - \, \mathcal{L} \, = \, 0$$

Reparametrization invariance

$$\tau \to \tau'(\tau) \implies d\tau' \sqrt{-\dot{x}^2(\tau')} \, = \, d\tau \sqrt{-\dot{x}^2(\tau)}$$

analogue of general coordinate invariance in 0+1 d

Introduce a Lagrange multiplier N

$$H \, = \, \frac{N}{2m} \, (\, p^2 \, + \, m^2 \,)$$

$$\{x^\mu, p_\nu\} \, = \, \delta^\mu_\nu \quad \Longrightarrow \quad \dot{x}^\mu \, = \, \{ \, x^\mu, H \, \} \, = \, \frac{N x^\mu}{\sqrt{-\dot{x}^2}} \quad \Longrightarrow \quad \dot{x}^2 \, = \, -N^2$$

N is arbitrary (gauge election)

N=1 $\Longrightarrow$ $\ddot{x}^\mu = 0$ $\Longrightarrow$ $\boxed{\dot{x}^\mu(\tau) \, = \, x_0 \, + \, \frac{p^\mu}{m} \, \tau}$

Polyakov action

Define a metric on the worldline

$$ds^2 = e^2(\tau)\,(d\tau)^2 = g_{\tau\tau}\,(d\tau)^2$$

$e(\tau)$ is the einbein

$$e = \sqrt{g_{\tau\tau}}, \qquad g^{\tau\tau} = e^{-2}(\tau)$$

$$\boxed{S = \frac{1}{2}\int d\tau\, e(\tau)\left[e^{-2}(\tau)\,\dot{x}^2 - m^2\right] = \frac{1}{2}\int d\tau\,\sqrt{\det g}\left[g^{\tau\tau}\,\partial_\tau x\cdot\partial_\tau x - m^2\right]}$$

EOM of e:

$$\delta S = -\frac{1}{2}\int d\tau\left[\frac{\dot{x}^2}{e^2} + m^2\right]\delta e \quad\Longrightarrow\quad e = \frac{1}{m}\sqrt{-\dot{x}^2}$$

EOM of x:

$$\delta S = -\int d\tau\,\partial_\tau\left[e^{-1}\,\dot{x}^\mu\right]\delta x_\mu \quad\Longrightarrow\quad \partial_\tau\left[e^{-1}\,\dot{x}^\mu\right] = 0$$

Using the value of e:

$$\partial_\tau \left[\frac{m\,\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \right] = 0$$

same as before!

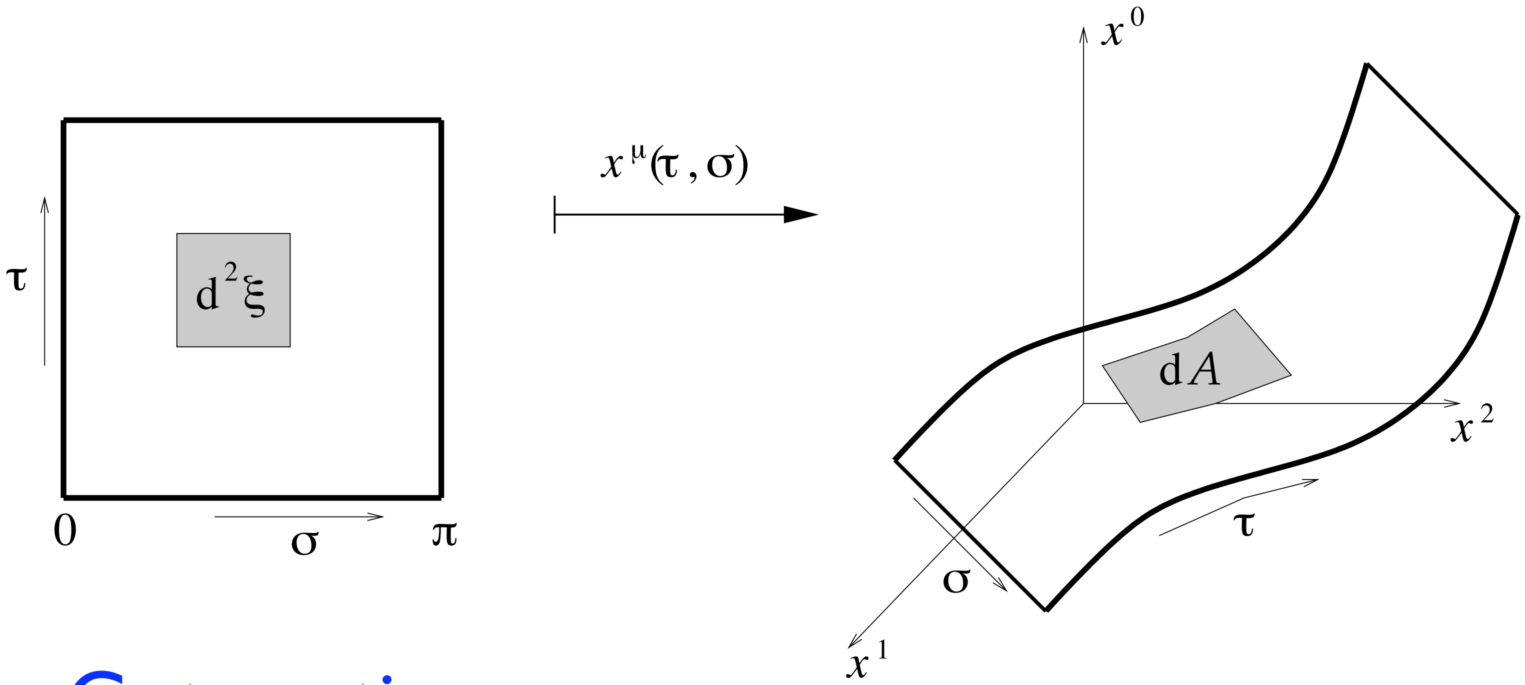
Substituting the value of e in the action:

$$S = \frac{1}{2} \int d\tau \left[e^{-1}(\tau)\,\dot{x}^2 - m^2\,e(\tau) \right] = -\int d\tau\, m\,\sqrt{-\dot{x}^2}$$

The Polyakov action reduces to the original action

The relativistic string

Extended object that sweeps a 2d worldsheet



Nambu-Goto action:

$$\boxed{S_{NG} = -T \int dA}$$

$$T = \frac{1}{2\pi\alpha'}$$

α' is the Regge slope

$$[\alpha'] = [L]^{-2} = [M]^2$$

Worldsheet coordinates

$$\xi^\alpha\,, \qquad \alpha = 0, 1\,, \qquad (\xi^0, \xi^1) = (\tau, \sigma)$$

$$\text{Embedding: } \Sigma \to \mathcal{M} \text{ with } \xi^a \to X^\mu(\xi^a)$$

Induced metric

$$ds^2 = G_{\mu\nu}\, dX^\mu(\xi)\, dX^\nu(\xi) = G_{\mu\nu}\, \frac{\partial X^\mu}{\partial \xi^\alpha}\, \frac{\partial X^\nu}{\partial \xi^\beta}\, d\xi^\alpha\, d\xi^\beta$$

$$\boxed{\hat{G}_{\alpha\beta} \equiv G_{\mu\nu}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu} \qquad ds^2 = \hat{G}_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta$$

Area element $\Rightarrow$ $dA = \sqrt{-\det \hat{G}_{\alpha\beta}}\; d^2\xi$

Action $\Rightarrow$ $\boxed{S_{NG} = -T \int \sqrt{-\det \hat{G}_{\alpha\beta}}\; d^2\xi}$

In a flat spacetime $G_{\mu\nu} = \eta_{\mu\nu}$

Induced metric

$$\det \hat{G}_{\alpha\beta} = \hat{G}_{00}\, \hat{G}_{11} - \hat{G}^2_{01} = \dot{X}^2\, X'^2 - (\dot{X} \cdot X')^2$$

$$X'^{\mu} \equiv \frac{\partial X^\mu}{\partial \sigma} \qquad \dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}$$

$$\Downarrow$$

$$\boxed{S_{NG} = -T \int \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2\, X'^2}\; d^2\xi \equiv \int \mathcal{L}\; d^2\xi}$$

EOM $\Rightarrow$

$$\partial_\tau \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \right) + \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial X'^{\mu}} \right) = 0$$

Non-linear PDE

Canonical analysis

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T \, \frac{(\dot{X} \cdot X')\, X'_\mu - X'^{\,2}\, \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2\, X'^2}}$$

Constraints

$$\Pi \cdot X' = 0 \qquad\qquad \Pi^2 + T^2 \left(X' \right)^2 = 0$$

Hamiltonian

$$H = \int d\sigma \left[\dot{X} \cdot \Pi - \mathcal{L} \right] = 0$$

The dynamics is governed by the constraints

Polyakov action

Introduce a worldsheet metric $g_{\alpha\beta}(\tau, \sigma)$

$$\boxed{S_P = -\frac{T}{2} \int d^2\xi \sqrt{-\det g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \eta_{\mu\nu}}$$

EOM of $g_{\alpha\beta}$: $$T_{\alpha\beta} \equiv -\frac{2}{T} \frac{1}{\sqrt{-\det g}} \frac{\delta S_P}{\delta g^{\alpha\beta}} \Longrightarrow \boxed{T_{\alpha\beta} = 0}$$

Since

$$\delta(\det g) = -\det g\, g_{\alpha\beta}\, \delta g^{\alpha\beta} \Longrightarrow \delta\left[\sqrt{-\det g}\right] = -\frac{1}{2} \sqrt{-\det g}\; g_{\alpha\beta}\, \delta g^{\alpha\beta}$$

Energy-momentum tensor

$$\boxed{T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} g_{\alpha\beta}\, g^{\gamma\delta}\, \partial_\gamma X \cdot \partial_\delta X}$$

From the eom of the 2d metric

$$\partial_\alpha X \cdot \partial_\beta X = \left(\frac{1}{2}\, g^{\gamma\delta}\, \partial_\gamma X \cdot \partial_\delta X\right) g_{\alpha\beta} \quad \Longrightarrow \quad \text{induced metric}$$

Substituting in the action

$$-T\sqrt{-\det(\partial_\alpha X \cdot \partial_\beta X)} = -\frac{T}{2}\, g^{\gamma\delta}\, \partial_\gamma X \cdot \partial_\delta X\, \sqrt{-\det g}$$

The Nambu-Goto and Polyakov actions coincide!!

EOM of X^μ

$$\boxed{\partial_\alpha \left[\sqrt{-\det g}\, g^{\alpha\beta}\, \partial_\beta X^\mu\right] = 0}$$

Space-time coordinates are scalar fields in 2d!!

Symmetries of the Polyakov action

Poincare transformations

$$\delta X^\mu = \omega^\mu{}_\nu\, X^\nu + a^\mu \ , \qquad \delta g_{\alpha\beta} = 0$$

$$\omega^\mu{}_\nu \text{ constant and } \omega_{\mu\nu} = -\omega_{\nu\mu}$$

2d reparametrizations

$$\xi^\alpha \to f^\alpha(\xi) = \xi'^{\,\alpha} \ , \qquad g_{\alpha\beta}(\xi) = \frac{\partial f^\gamma}{\partial \xi^\alpha} \frac{\partial f^\delta}{\partial \xi^\beta}\, g_{\gamma\delta}(\xi')$$

Local Weyl transformations

$$g_{\alpha\beta} \to e^{\Phi(\tau,\sigma)}\, g_{\alpha\beta} \ , \qquad \delta X^\mu = 0$$

The action is invariant because

$$\sqrt{-\det g} \to e^{\Phi} \sqrt{-\det g} \ , \qquad g^{\alpha\beta} \to e^{-\Phi} g^{\alpha\beta}$$

The ws metric $g_{\alpha\beta}$ has three independent components:

$$\begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} , \qquad\qquad g_{01} = g_{10}$$

With a reparametrization we can make the metric conformally flat

$$g_{\alpha\beta} \;=\; e^{\Lambda}\,\eta_{\alpha\beta} \;, \qquad\qquad \eta_{\alpha\beta} \;=\; \mathrm{diag}(-1,+1)$$

Using Weyl invariance

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Polyakov action can be gauge-fixed to:

$$\boxed{S_P \;=\; -\frac{T}{2} \int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma \; \eta^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu}$$

EOM for the flat ws metric

$$\partial_\alpha \, \partial^\alpha \, X^\mu \, = \, 0$$

Explicitly

$$\boxed{\left[\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right] X^\mu \, = \, 0}$$

just Klein-Gordon in 2d !

The constraints are

$$T_{00} \, = \, T_{11} \, = \, \frac{1}{2} \left(\dot{X}^2 \, + \, X'^2 \right) \, = \, 0 \; , \qquad T_{01} \, = \, T_{10} \, = \, \dot{X} \cdot X' \, = \, 0$$

They can be rewritten as:

$$\boxed{\left(\dot{X} \pm X' \right)^2 \, = \, 0}$$

Noether currents

Consider a symmetry transformation

$$\phi \rightarrow \phi + \delta_\epsilon \, \phi$$

Taking $\epsilon = \epsilon(\xi)$ $$\mathcal{L} \rightarrow \mathcal{L} + \epsilon \, \partial_\alpha \, J^\alpha$$

Translations: $$X^\mu \rightarrow X^\mu + b^\mu$$

$$\boxed{P^\mu_\alpha = T \, \partial_\alpha \, X^\mu}$$

Conserved current associated to Poincare invariance

Associated charge $\Longrightarrow$ $$\boxed{p^\mu = \int_0^{\bar{\sigma}} P^\mu_\tau \, d\sigma}$$ Momentum

Lorentz transformation

$$\delta X^{\mu} = \omega^{\mu}_{\ \nu} X^{\nu} \qquad\qquad \omega^{\mu}_{\ \nu} \text{ constant and } \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Conserved current

$$\boxed{J^{\mu\nu}_{\alpha} = T \left(X^{\mu} \partial_{\alpha} X^{\nu} - X^{\nu} \partial_{\alpha} X^{\mu} \right)}$$

Generator of Lorentz transformations

$$J^{\mu\nu} = \int d\sigma \, J^{\mu\nu}_{\tau} = T \int d\sigma \left[X^{\mu} \dot{X}^{\nu} - X^{\nu} \dot{X}^{\mu} \right]$$

Boundary conditions

Let us compute a general variation of the action:

$$\delta S_P = T \int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma \, \partial^\alpha \partial_\alpha X^\mu \, \delta X_\mu - T \int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma \, \partial_\alpha \left[\partial^\alpha X^\mu \, \delta X_\mu \right]$$

Since $\delta X^\mu \big|_{\tau=\tau_{1,2}} = 0$ in the variational procedure:

$$\boxed{\delta S_P = T \int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma \, \partial^\alpha \partial_\alpha X^\mu \, \delta X_\mu - T \int_{\tau_1}^{\tau_2} d\tau \left[X'^\mu \, \delta X_\mu \right]_{\sigma=0}^{\sigma=\bar{\sigma}}}$$

The vanishing of the first term implies the eom of X

We would require also

$$\boxed{X'^{\mu}\,\delta X_{\mu}\Big|_{\sigma=0}^{\sigma=\bar{\sigma}} = 0}$$

For closed strings

Take $\bar{\sigma} = 2\pi$ and impose periodic boundary conditions

$$X^{\mu}(\tau, \sigma + 2\pi) = X^{\mu}(\tau, \sigma)$$

The boundary requirement is automatically satisfied

Open strings

Take $\bar{\sigma} = \pi$

There are two possibilities

- Neumann boundary conditions

$$\boxed{X'(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0}$$

- Dirichlet boundary conditions

$$\delta X(\sigma = 0, \pi) = 0 \quad \Longleftrightarrow \quad \dot{X}(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0$$

X is fixed at $\sigma = 0, \pi$

$$\boxed{\begin{aligned} X^{\mu}\Big|_{\sigma=\pi} &= X^{\mu}_{\pi} \\ X^{\mu}\Big|_{\sigma=0} &= X^{\mu}_{0} \end{aligned}}$$

breaks Poincare invariance

Momentum flow

$$p_\mu = \int_0^{\bar\sigma} P^\tau_\mu \, d\sigma \quad \Rightarrow \quad \frac{dp_\mu}{d\tau} = \int_0^{\bar\sigma} \frac{dP^\tau_\mu}{d\tau} \, d\sigma$$

Since:

$$P^\tau_\mu = T \partial_\tau X_\mu \quad \Rightarrow \quad \frac{dP^\tau_\mu}{d\tau} = T \, \partial^2_\tau X_\mu = T \, \partial^2_\sigma X_\mu$$

Total momentum variation

$$\boxed{\frac{dp_\mu}{d\tau} = T \int_0^{\bar\sigma} \partial^2_\sigma X_\mu \, d\sigma = T \left[X'_\mu(\tau, \sigma = \bar\sigma) - X'_\mu(\tau, \sigma = 0) \right]}$$

The momentum flow vanishes for:

- Closed strings
- Open strings with Neumann boundary conditions

(No momentum flow off the ends of the string)

Free boundary conditions

Open strings with Dirichlet boundary conditions end on some object that absorbs the momentum

$\Downarrow$

D-branes