String Theory

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Lecture 2

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

Alfonso V. Ramallo

The rotating open string solution

$$X^0 = B\tau \ , \qquad X^1 = B\cos\tau\cos\sigma \ , \qquad X^2 = B\sin\tau\cos\sigma$$

represents a string rotating in the X^1X^2 plane

satisfies the eom&constraints

$$\dot{X}^\mu \pm X^{\mu\prime} = B\left(1, -\sin(\tau\pm\sigma), \cos(\tau\pm\sigma)\right) \Longrightarrow \left(\dot{X} \pm X'\right)^2 = 0$$

Energy

$\Longrightarrow$ $$E = p^0 = T\int_0^\pi \dot{X}^0 = \pi T B$$

Momentum

$$J = J^{12} = T\int_0^\pi d\sigma\left[X^1\dot{X}^2 - X^2\dot{X}^1\right] = TB^2\int_0^\pi d\sigma\ \cos^2\sigma = \frac{\pi T B^2}{2}$$

$$\frac{E^2}{J} = 2\pi T \Longrightarrow \boxed{J = \alpha' E^2}$$

α' is the Regge slope

Light-cone coordinates of the worldsheet

$$\xi^+ = \tau + \sigma \qquad \xi^- = \tau - \sigma$$

WS metric

$$ds^2 = -d\xi^+ \, d\xi^-$$

$$g_{++} = g_{--} = 0 \qquad g_{+-} = g_{-+} = -\frac{1}{2}$$

Derivatives

$$\partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right) \Rightarrow \partial_\pm \, \xi^\pm = 1 \qquad \partial_\pm \, \xi^\mp = 0$$

Similar to holomorphic and antiholomorphic variables in the complex plane

Polyakov action

$$S_P = 2T \int d^2\xi \, \partial_+ X^\mu \, \partial_- X^\nu \, \eta_{\mu\nu}$$

EOM

$$\partial_+ \partial_- X^\mu = 0$$

Solution

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$$

Mode expansions

Right movers

$$X_R^\mu(\tau-\sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2}\, p^\mu\,(\tau-\sigma) + \frac{i\,l_s}{\sqrt{2}} \sum_{k\neq 0} \frac{\alpha_k^\mu}{k}\, e^{-ik(\tau-\sigma)}$$

Left movers

$$X_L^\mu(\tau+\sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2}\, \bar{p}^\mu\,(\tau+\sigma) + \frac{i\,l_s}{\sqrt{2}} \sum_{k\neq 0} \frac{\bar{\alpha}_k^\mu}{k}\, e^{-ik(\tau+\sigma)}$$

$X_L,\, X_R$ real $\rightarrow$ x^μ , p^μ , $\bar{p}^\mu$ real and

$$(\alpha_k^\mu)^* = \alpha_{-k}^\mu\,, \qquad (\bar{\alpha}_k^\mu)^* = \bar{\alpha}_{-k}^\mu$$

The α's and $\bar{\alpha}$'s are oscillator variables

Closed strings

$$X(\tau,\sigma+2\pi) = X(\tau,\sigma) \implies k \in \mathbb{Z}-\{0\}\ ,\ p^\mu = \bar{p}^\mu$$

$$X_R^\mu(\tau-\sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2}\, p^\mu\,(\tau-\sigma) + \frac{i\,l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}-\{0\}} \frac{\alpha_n^\mu}{n}\, e^{-in(\tau-\sigma)}$$

$$X_L^\mu(\tau+\sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2}\, p^\mu\,(\tau+\sigma) + \frac{i\,l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}-\{0\}} \frac{\bar{\alpha}_n^\mu}{n}\, e^{-in(\tau+\sigma)}$$

Define →

$$\alpha_0^\mu = \frac{l_s}{\sqrt{2}}\, p^\mu = \bar{\alpha}_0^\mu$$

$$\partial_-\, X_R = \frac{l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu\, e^{-in(\tau-\sigma)}$$

$$\partial_+\, X_L = \frac{l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu\, e^{-in(\tau+\sigma)}$$

Center of mass coordinate

$$x^{\mu}_{CM} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \, X^{\mu}(\tau, \sigma) = x^{\mu} + l_s^2 \, p^{\mu} \, \tau$$

x^{μ} is the center of mass of the string at $\tau = 0$, which moves as a free particle

Center of mass momentum

$$p^{\mu}_{CM} = T \int_0^{2\pi} d\sigma \, \dot{X}^{\mu} = T \, l_s^2 \, p^{\mu} \int_0^{2\pi} d\sigma = p^{\mu}$$

p^{μ} is the center of mass momentum

Virasoro constraints

EM tensor in light-cone coordinates

$$T_{++} = \partial_+ X \cdot \partial_+ X \,, \qquad T_{--} = \partial_- X \cdot \partial_- X$$

$$T_{+-} = T_{-+} = 0$$

Fourier transform of T_{++} and T_{--}

$$L_m = T \int_0^{2\pi} d\sigma \, T_{--} \, e^{im(\tau-\sigma)} \,, \qquad \bar{L}_m = T \int_0^{2\pi} d\sigma \, T_{++} \, e^{im(\tau+\sigma)}$$

Inverse relations → **use** $\int_0^{2\pi} e^{in\sigma} \, d\sigma = 2\pi \, \delta_n$

$$T_{--} = l_s^2 \sum_{m=-\infty}^{+\infty} L_m \, e^{-im(\tau-\sigma)} \,, \qquad T_{++} = l_s^2 \sum_{m=-\infty}^{+\infty} \bar{L}_m \, e^{-im(\tau+\sigma)}$$

L_m and $\bar{L}_m$ are the Virasoro generators

Mode expansion of Virasoro generators:

$$L_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \ , \qquad \bar{L}_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n$$

Reality conditions

$$L_m^* = L_{-m} \ , \qquad \bar{L}_m^* = \bar{L}_{-m}$$

Constraints in terms of Fourier modes

$$L_m = 0 \ , \qquad \bar{L}_m = 0 \ , \qquad m = 0, \pm 1, \pm 2, \cdots$$

Classical mass formulae

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$$

$$\alpha_0^\mu = \frac{l_s}{\sqrt{2}} p^\mu$$

$$\Downarrow$$

$$\alpha_0^2 = -\frac{l_s^2}{2} M^2$$

From $L_0 = 0$ $\Rightarrow$

$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

Similarly from $\bar{L}_0 = 0$

$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{+\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n$$

For consistency

$$\sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{+\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n$$

Mass formula in symmetric form

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{+\infty} \left(\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \right)$$

continuous mass spectra at the classical level

Hamiltonian analysis of the closed string

Canonical momentum $\Longrightarrow$ $\Pi^\mu(\tau,\sigma) = \frac{\delta}{\delta \dot{X}_\mu(\tau,\sigma)} S_P$

$$\mathcal{L} = \frac{T}{2}(\dot{X}^2 - X'^2) \Longrightarrow \Pi^\mu = T\,\dot{X}^\mu$$

Hamiltonian $\Longrightarrow$ $H = \int_0^{2\pi} \left(\dot{X}_\mu \Pi^\mu - \mathcal{L} \right)$

$$H = \frac{T}{2}\int_0^{2\pi} (\dot{X}^2 + X'^2)$$

In terms of the EM tensor

$$H = T\int_0^{2\pi} \left[T_{++} + T_{--} \right] \quad \Longrightarrow \quad H = L_0 + \bar{L}_0$$

canonical equal-time Poisson brackets

$$\left[X^\mu(\tau,\sigma)\,,\, X^\nu(\tau,\sigma') \right]_{PB} = 0$$

$$\left[\Pi^\mu(\tau,\sigma)\,,\, \Pi^\nu(\tau,\sigma') \right]_{PB} = 0$$

$$\left[\Pi^\mu(\tau,\sigma)\,,\, X^\nu(\tau,\sigma') \right]_{PB} = \eta^{\mu\nu}\delta(\sigma-\sigma')$$

As $\Pi^\mu = T\,\dot{X}^\mu$ the only non-trivial PB is:

$$\left[\,\dot{X}^\mu(\tau,\sigma)\,,\,X^\nu(\tau,\sigma')\,\right]_{PB} = T^{-1}\,\eta^{\mu\nu}\delta(\sigma-\sigma')$$

In terms of the modes these PB are equivalent to:

$$\left[\,p^\mu, x^\nu\,\right]_{PB} = \eta^{\mu\nu}$$

$$\left[\,\alpha^\mu_m\,,\,\alpha^\nu_n\,\right]_{PB} = im\,\delta_{m+n}\,\eta^{\mu\nu}$$

$$\left[\,\bar{\alpha}^\mu_m\,,\,\bar{\alpha}^\nu_n\,\right]_{PB} = im\,\delta_{m+n}\,\eta^{\mu\nu}$$

To prove the last result we should use

$$\delta(\sigma - \sigma') = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\sigma - \sigma')}$$

which follows from the orthogonality and completeness of the wave functions

$$< \sigma | n > = \frac{1}{\sqrt{2\pi}} e^{in\sigma}$$

Check:

$$\delta(\sigma - \sigma') = < \sigma | \sigma' > = \sum_{n \in \mathbb{Z}} < \sigma | n > < n | \sigma' > = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\sigma - \sigma')}$$

Poisson algebra of the Virasoro generators

Use:

$$[AB\,,CD]_{PB} = A\,[B\,,C]_{PB}\,D + AC\,[B\,,D]_{PB} + $$

$$+ [A\,,C]_{PB}\,DB + C\,[A\,,D]_{PB}\,B$$

⇓

$$[L_m\,,L_n]_{PB} = \frac{1}{4}\sum_{k,l}\left[\alpha_{m-k}\cdot\alpha_k\,,\alpha_{n-l}\cdot\alpha_l\right]_{PB} =$$

$$= \frac{i}{4}\sum_{k,l}\Big[k\alpha_{m-k}\cdot\alpha_l\;\delta_{k+n-l} + k\alpha_{m-k}\cdot\alpha_{n-l}\;\delta_{k+l} +$$

$$+ (m-k)\alpha_l\cdot\alpha_k\;\delta_{m-k+n-l} + (m-k)\alpha_{n-l}\cdot\alpha_k\;\delta_{m-k+l}\Big]$$

⇓

$$[L_m\,,L_n]_{PB} = \frac{i}{2}\sum_k k\,\alpha_{m-k}\cdot\alpha_{k+n} + \frac{i}{2}\sum_k (m-k)\,\alpha_{m-k+n}\cdot\alpha_k$$

$$\left[\, L_m \,,\, L_n \,\right]_{PB} = i(n-m)\, L_{m+n}$$ Classical Virasoro algebra

The origin of the Virasoro algebra is the residual symmetry

$$\xi^{\pm} \to \xi^{\pm\prime} = F_{\pm}(\xi_{\pm})$$

$$d^2\xi' = \partial_+ F_+ \partial_- F_-\, d^2\xi$$

$$\partial'_{\pm} X^{\mu} = \frac{\partial X^{\mu}}{\partial \xi^{\pm}} \frac{\partial \xi^{\pm}}{\partial \xi^{\pm\prime}} = \frac{1}{\partial_{\pm} F_{\pm}}\, \partial_{\pm} X^{\mu}$$

Then

$$d^2\xi'\, \partial'_+ X \cdot \partial'_- X = d^2\xi\, \partial_+ X \cdot \partial_- X$$ ⇒ the action is invariant

Consider the operators

$$V^{\pm} = f_{\pm}(\xi^{\pm}) \frac{\partial}{\partial \xi^{\pm}}$$

They generate $\delta\xi^{\pm} = f_{\pm}$

Take $f_n^{\pm}(\xi^{\pm}) = e^{in\xi^{\pm}}\,, n \in \mathbb{Z}$ ⇒ $V_n^{\pm} = e^{in\xi^{\pm}} \dfrac{\partial}{\partial \xi^{\pm}}$

They satisfy ⇒ $\left[\, V_n^{\pm} \,,\, V_m^{\pm} \,\right] = i(n-m)\, V_{n+m}^{\pm}$

Open Strings with NN boundary conditions

Take $\bar{\sigma} = \pi$ $\qquad X^{\mu\prime}(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0$

$$X^{\mu\prime}(\tau,\sigma) = \frac{l_s^2}{2}\left(\bar{p}^\mu - p^\mu\right) - \frac{l_s}{\sqrt{2}}\sum_{k\neq 0}\left[\alpha_k^\mu\, e^{-ik(\tau-\sigma)} - \bar{\alpha}_k^\mu\, e^{-ik(\tau+\sigma)}\right]$$

$$X^{\mu\prime}(\tau,\sigma=0) = \frac{l_s^2}{2}\left(\bar{p}^\mu - p^\mu\right) - \frac{l_s}{\sqrt{2}}\sum_{k\neq 0} e^{-ik\tau}\left(\alpha_k^\mu - \bar{\alpha}_k^\mu\right)$$

Then

$$p^\mu = \bar{p}^\mu\ , \qquad \alpha_k^\mu = \bar{\alpha}_k^\mu$$

Only one set of oscillators!

$$X^{\mu\prime}(\tau,\sigma) = -i\sqrt{2}\,l_s\sum_{k\neq 0}\alpha_k^\mu\,\sin k\sigma\, e^{-ik\tau}$$

$X^{\mu\prime}(\tau,\sigma=\pi) = 0$ implies that $k \in \mathbb{Z} - \{0\}$

Changing $p^\mu \to 2p^\mu$

$$X^\mu(\tau,\sigma) \;=\; x^\mu \;+\; l_s^2\, p^\mu\, \tau \;+\; i\sqrt{2}\, l_s \sum_{n\in\mathbb{Z}-\{0\}} \frac{\alpha_n^\mu}{n}\, e^{-in\tau}\, \cos n\sigma$$

Define

$$\alpha_0 \;=\; \sqrt{2}\, l_s\, p^\mu \quad \Rightarrow \quad \partial_\pm X^\mu \;=\; \frac{l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu\, e^{-in(\tau\pm\sigma)}$$

CM position&momentum

$$X_{CM}^\mu \;=\; \frac{1}{\pi}\int_0^\pi d\sigma X^\mu(\tau,\sigma) \;=\; x^\mu \;+\; 2l_s^2\, p^\mu\, \tau$$

$$P_{CM}^\mu \;=\; T\int_0^\pi d\sigma\, \dot{X}^\mu \;=\; p^\mu$$

Hamiltonian

$$H_{NN} \;=\; \frac{T}{2}\int_0^\pi d\sigma\left(\dot{X}^2 \;+\; X'^2\right) \;=\; T\int_0^\pi d\sigma\left[(\partial_+X)^2 \;+\; (\partial_-X)^2\right]$$

Mode expansion

$$H_{NN} = \frac{1}{2} \sum_{n\in\mathbb{Z}} \alpha_{-n} \cdot \alpha_n \quad \Rightarrow \quad H_{NN} = l_s^2\, p^2 + \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

Define

$$L_m = T \int_0^{\pi} d\sigma \left[T_{++}\, e^{in(\tau+\sigma)} + T_{--}\, e^{im(\tau-\sigma)} \right]$$

mode expansion

$$L_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \qquad H_{NN} = L_0$$

Classical mass formula

$$m^2 = \frac{1}{\alpha'} \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

Open strings with DD boundary conditions

$$\dot{X}^\mu\,(\tau,\sigma)\,\Big|_{\sigma=0,\pi} \;=\; 0$$

$$\Downarrow$$

$$\dot{X}^\mu(\tau,\sigma) \;=\; \frac{l_s^2}{2}\,\left(p^\mu + \bar{p}^\mu\right) \;-\; \frac{l_s}{\sqrt{2}}\sum_{k\neq 0}\left[\alpha_k^\mu\, e^{-ik(\tau-\sigma)} \;+\; \bar{\alpha}_k^\mu\, e^{-ik(\tau+\sigma)}\right]$$

$$\Downarrow$$

$$\dot{X}^\mu(\tau,\sigma=0) \;=\; \frac{l_s^2}{2}\,\left(p^\mu + \bar{p}^\mu\right) \;+\; \frac{l_s}{\sqrt{2}}\sum_{k\neq 0}\; e^{-ik(\tau)}(\alpha_k^\mu \;+\; \bar{\alpha}_k^\mu\,)$$

Then

$$p^\mu = -\bar{p}^\mu\;, \qquad \alpha_k^\mu \;=\; -\bar{\alpha}_k^\mu$$

$$\dot{X}^\mu(\tau,\sigma) \;=\; -i\sqrt{2}\,l_s\sum_{k\neq 0}\;\bar{\alpha}_k^\mu\;\sin k\sigma\, e^{-ik\tau}$$

$$\dot{X}^\mu\,(\tau,\sigma)\,\Big|_{\pi} \;=\; 0 \text{ implies } k\in\mathbb{Z}$$

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2\,\bar{p}^\mu\,\sigma + \sqrt{2}\,l_s \sum_{n\in\mathbb{Z}-\{0\}} \frac{\bar{\alpha}_n^\mu}{n}\,e^{-in\tau}\,\sin n\sigma$$

Change $\bar{\alpha}_n^\mu \to \alpha_n^\mu$

As $X_0^\mu = x^\mu\,,\qquad X_\pi^\mu = x^\mu + \pi l_s^2\,\bar{p}^\mu$

$$\Downarrow$$

$$X^\mu(\tau,\sigma) = X_0^\mu + \frac{X_\pi^\mu - X_0^\mu}{\pi}\,\sigma + \sqrt{2}\,l_s \sum_{n\in\mathbb{Z}-\{0\}} \frac{\alpha_n^\mu}{n}\,e^{-in\tau}\,\sin n\sigma$$