

STrÍng Theory



Lecture 2

Master en Física Nuclear e de Partículas e
as súas aplicacións Tecnolóxicas e Médicas

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The rotating open string solution

$$X^0 = B\tau, \quad X^1 = B \cos \tau \cos \sigma, \quad X^2 = B \sin \tau \cos \sigma$$

represents a string rotating in the $X^1 X^2$ plane

satisfies the eom&constraints

$$\dot{X}^\mu \pm X^{\mu'} = B (1, -\sin(\tau \pm \sigma), \cos(\tau \pm \sigma)) \implies (\dot{X} \pm X')^2 = 0$$

Energy $\implies E = p^0 = T \int_0^\pi \dot{X}^0 = \pi T B$

Momentum

$$J = J^{12} = T \int_0^\pi d\sigma [X^1 \dot{X}^2 - X^2 \dot{X}^1] = T B^2 \int_0^\pi d\sigma \cos^2 \sigma = \frac{\pi T B^2}{2}$$

$$\frac{E^2}{J} = 2\pi T \implies \boxed{J = \alpha' E^2} \quad \alpha' \text{ is the Regge slope}$$

Light-cone coordinates of the worldsheet

$$\xi^+ = \tau + \sigma \qquad \xi^- = \tau - \sigma$$

WS metric

$$ds^2 = -d\xi^+ d\xi^-$$

$$g_{++} = g_{--} = 0 \qquad g_{+-} = g_{-+} = -\frac{1}{2}$$

Derivatives

$$\partial_{\pm} = \frac{1}{2} (\partial_{\tau} \pm \partial_{\sigma}) \quad \longrightarrow \quad \partial_{\pm} \xi^{\pm} = 1 \quad \partial_{\pm} \xi^{\mp} = 0$$

Similar to holomorphic and antiholomorphic variables in the complex plane

Polyakov action

$$S_P = 2T \int d^2\xi \partial_+ X^\mu \partial_- X^\nu \eta_{\mu\nu}$$

EOM

$$\partial_+ \partial_- X^\mu = 0$$

Solution

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$$

Mode expansions

Right movers

$$X_R^\mu(\tau - \sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2} p^\mu (\tau - \sigma) + \frac{i l_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\alpha_k^\mu}{k} e^{-ik(\tau - \sigma)}$$

Left movers

$$X_L^\mu(\tau + \sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2} \bar{p}^\mu (\tau + \sigma) + \frac{i l_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\bar{\alpha}_k^\mu}{k} e^{-ik(\tau + \sigma)}$$

X_L, X_R real $\rightarrow x^\mu, p^\mu, \bar{p}^\mu$ real and

$$(\alpha_k^\mu)^* = \alpha_{-k}^\mu, \quad (\bar{\alpha}_k^\mu)^* = \bar{\alpha}_{-k}^\mu$$

The α 's and $\bar{\alpha}$'s are oscillator variables

Closed strings

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) \implies k \in \mathbb{Z} - \{0\}, \quad p^\mu = \bar{p}^\mu$$

$$X_R^\mu(\tau - \sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2} p^\mu (\tau - \sigma) + \frac{i l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z} - \{0\}} \frac{\alpha_n^\mu}{n} e^{-in(\tau - \sigma)}$$

$$X_L^\mu(\tau + \sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2} p^\mu (\tau + \sigma) + \frac{i l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z} - \{0\}} \frac{\bar{\alpha}_n^\mu}{n} e^{-in(\tau + \sigma)}$$

Define \longrightarrow $\alpha_0^\mu = \frac{l_s}{\sqrt{2}} p^\mu = \bar{\alpha}_0^\mu$

$$\partial_- X_R = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in(\tau - \sigma)}$$

$$\partial_+ X_L = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \bar{\alpha}_n^\mu e^{-in(\tau + \sigma)}$$

Center of mass coordinate

$$x_{CM}^{\mu} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma X^{\mu}(\tau, \sigma) = x^{\mu} + l_s^2 p^{\mu} \tau$$

x^{μ} is the center of mass of the string at $\tau = 0$, which moves as a free particle

Center of mass momentum

$$p_{CM}^{\mu} = T \int_0^{2\pi} d\sigma \dot{X}^{\mu} = T l_s^2 p^{\mu} \int_0^{2\pi} d\sigma = p^{\mu}$$

p^{μ} is the center of mass momentum

Virasoro constraints

EM tensor in light-cone coordinates

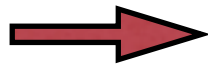
$$T_{++} = \partial_+ X \cdot \partial_+ X , \quad T_{--} = \partial_- X \cdot \partial_- X$$

$$T_{+-} = T_{-+} = 0$$

Fourier transform of T_{++} and T_{--}

$$L_m = T \int_0^{2\pi} d\sigma T_{--} e^{im(\tau-\sigma)} , \quad \bar{L}_m = T \int_0^{2\pi} d\sigma T_{++} e^{im(\tau+\sigma)}$$

Inverse relations



use

$$\int_0^{2\pi} e^{in\sigma} d\sigma = 2\pi \delta_n$$

$$T_{--} = l_s^2 \sum_{m=-\infty}^{+\infty} L_m e^{-im(\tau-\sigma)} , \quad T_{++} = l_s^2 \sum_{m=-\infty}^{+\infty} \bar{L}_m e^{-im(\tau+\sigma)}$$

L_m and \bar{L}_m are the Virasoro generators

Mode expansion of Virasoro generators:

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n, \quad \bar{L}_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n$$

Reality conditions

$$L_m^* = L_{-m}, \quad \bar{L}_m^* = \bar{L}_{-m}$$

Constraints in terms of Fourier modes

$$L_m = 0, \quad \bar{L}_m = 0, \quad m = 0, \pm 1, \pm 2, \dots$$

Classical mass formulae

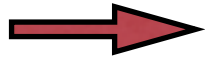
$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$$

$$\alpha_0^\mu = \frac{l_s}{\sqrt{2}} p^\mu$$

↓

$$\alpha_0^2 = -\frac{l_s^2}{2} M^2$$

From $L_0 = 0$



$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

Similarly from $\bar{L}_0 = 0$

$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{+\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n$$

For consistency

$$\sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{+\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n$$

Mass formula in symmetric form

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{+\infty} \left(\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \right)$$

continuous mass spectra at the classical level

Hamiltonian analysis of the closed string

Canonical momentum $\longrightarrow \Pi^\mu(\tau, \sigma) = \frac{\delta}{\delta \dot{X}_\mu(\tau, \sigma)} S_P$

$$\mathcal{L} = \frac{T}{2} (\dot{X}^2 - X'^2) \longrightarrow \Pi^\mu = T \dot{X}^\mu$$

Hamiltonian $\longrightarrow H = \int_0^{2\pi} \left(\dot{X}_\mu \Pi^\mu - \mathcal{L} \right)$

$$H = \frac{T}{2} \int_0^{2\pi} (\dot{X}^2 + X'^2)$$

In terms of the EM tensor

$$H = T \int_0^{2\pi} [T_{++} + T_{--}] \longrightarrow H = L_0 + \bar{L}_0$$

canonical equal-time Poisson brackets

$$[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')]_{PB} = 0$$

$$[\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')]_{PB} = 0$$

$$[\Pi^\mu(\tau, \sigma), X^\nu(\tau, \sigma')]_{PB} = \eta^{\mu\nu} \delta(\sigma - \sigma')$$

As $\Pi^\mu = T \dot{X}^\mu$ the only non-trivial PB is:

$$[\dot{X}^\mu(\tau, \sigma), X^\nu(\tau, \sigma')]_{PB} = T^{-1} \eta^{\mu\nu} \delta(\sigma - \sigma')$$

In terms of the modes these PB are equivalent to:

$$[p^\mu, x^\nu]_{PB} = \eta^{\mu\nu}$$

$$[\alpha_m^\mu, \alpha_n^\nu]_{PB} = im \delta_{m+n} \eta^{\mu\nu}$$

$$[\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu]_{PB} = im \delta_{m+n} \eta^{\mu\nu}$$

To prove the last result we should use

$$\delta(\sigma - \sigma') = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\sigma - \sigma')}$$

which follows from the orthogonality and completeness of the wave functions

$$\langle \sigma | n \rangle = \frac{1}{\sqrt{2\pi}} e^{in\sigma}$$

Check:

$$\delta(\sigma - \sigma') = \langle \sigma | \sigma' \rangle = \sum_{n \in \mathbb{Z}} \langle \sigma | n \rangle \langle n | \sigma' \rangle = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\sigma - \sigma')}$$

Poisson algebra of the Virasoro generators

Use:

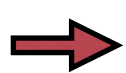
$$\begin{aligned} [AB, CD]_{PB} &= A [B, C]_{PB} D + AC [B, D]_{PB} + \\ &\quad + [A, C]_{PB} DB + C [A, D]_{PB} B \end{aligned}$$



$$\begin{aligned} [L_m, L_n]_{PB} &= \frac{1}{4} \sum_{k,l} [\alpha_{m-k} \cdot \alpha_k, \alpha_{n-l} \cdot \alpha_l]_{PB} = \\ &= \frac{i}{4} \sum_{k,l} \left[k \alpha_{m-k} \cdot \alpha_l \delta_{k+n-l} + k \alpha_{m-k} \cdot \alpha_{n-l} \delta_{k+l} + \right. \\ &\quad \left. + (m-k) \alpha_l \cdot \alpha_k \delta_{m-k+n-l} + (m-k) \alpha_{n-l} \cdot \alpha_k \delta_{m-k+l} \right] \end{aligned}$$



$$[L_m, L_n]_{PB} = \frac{i}{2} \sum_k k \alpha_{m-k} \cdot \alpha_{k+n} + \frac{i}{2} \sum_k (m-k) \alpha_{m-k+n} \cdot \alpha_k$$



$$\left[L_m, L_n \right]_{PB} = i(n - m) L_{m+n}$$

Classical Virasoro algebra

The origin of the Virasoro algebra is the residual symmetry

$$\xi^\pm \rightarrow \xi^{\pm'} = F_\pm(\xi_\pm)$$

$$d^2 \xi' = \partial_+ F_+ \partial_- F_- d^2 \xi$$

$$\partial'_\pm X^\mu = \frac{\partial X^\mu}{\partial \xi^\pm} \frac{\partial \xi^\pm}{\partial \xi^{\pm'}} = \frac{1}{\partial_\pm F_\pm} \partial_\pm X^\mu$$

Then

$$d^2 \xi' \partial'_+ X \cdot \partial'_- X = d^2 \xi \partial_+ X \cdot \partial_- X \quad \Rightarrow \quad \text{the action is invariant}$$

Consider the operators

$$V^\pm = f_\pm(\xi^\pm) \frac{\partial}{\partial \xi^\pm}$$

They generate $\delta \xi^\pm = f_\pm$

Take $f_n^\pm(\xi^\pm) = e^{in\xi^\pm}, n \in \mathbb{Z} \quad \Rightarrow \quad V_n^\pm = e^{in\xi^\pm} \frac{\partial}{\partial \xi^\pm}$

They satisfy $\Rightarrow [V_n^\pm, V_m^\pm] = i(n - m) V_{n+m}^\pm$

Open Strings with NN boundary conditions

Take $\bar{\sigma} = \pi$ $X^{\mu'}(\tau, \sigma) \big|_{\sigma=0, \pi} = 0$

$$X^{\mu'}(\tau, \sigma) = \frac{l_s^2}{2} (\bar{p}^\mu - p^\mu) - \frac{l_s}{\sqrt{2}} \sum_{k \neq 0} \left[\alpha_k^\mu e^{-ik(\tau-\sigma)} - \bar{\alpha}_k^\mu e^{-ik(\tau+\sigma)} \right]$$

$$X^{\mu'}(\tau, \sigma = 0) = \frac{l_s^2}{2} (\bar{p}^\mu - p^\mu) - \frac{l_s}{\sqrt{2}} \sum_{k \neq 0} e^{-ik\tau} (\alpha_k^\mu - \bar{\alpha}_k^\mu)$$

Then

$$p^\mu = \bar{p}^\mu, \quad \alpha_k^\mu = \bar{\alpha}_k^\mu$$

Only one set of oscillators!

$$X^{\mu'}(\tau, \sigma) = -i\sqrt{2}l_s \sum_{k \neq 0} \alpha_k^\mu \sin k\sigma e^{-ik\tau}$$

$$X^{\mu'}(\tau, \sigma = \pi) = 0 \text{ implies that } k \in \mathbb{Z} - \{0\}$$

Changing $p^\mu \rightarrow 2p^\mu$

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 p^\mu \tau + i\sqrt{2} l_s \sum_{n \in \mathbb{Z} - \{0\}} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma$$

Define

$$\alpha_0 = \sqrt{2} l_s p^\mu \quad \Rightarrow \quad \partial_\pm X^\mu = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in(\tau \pm \sigma)}$$

CM position&momentum

$$X_{CM}^\mu = \frac{1}{\pi} \int_0^\pi d\sigma X^\mu(\tau, \sigma) = x^\mu + 2l_s^2 p^\mu \tau$$

$$P_{CM}^\mu = T \int_0^\pi d\sigma \dot{X}^\mu = p^\mu$$

Hamiltonian

$$H_{NN} = \frac{T}{2} \int_0^\pi d\sigma (\dot{X}^2 + X'^2) = T \int_0^\pi d\sigma [(\partial_+ X)^2 + (\partial_- X)^2]$$

Mode expansion

$$H_{NN} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_n \quad \Rightarrow \quad H_{NN} = l_s^2 p^2 + \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

Define

$$L_m = T \int_0^\pi d\sigma \left[T_{++} e^{in(\tau+\sigma)} + T_{--} e^{im(\tau-\sigma)} \right]$$

mode expansion

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n$$

$$H_{NN} = L_0$$

Classical mass formula

$$m^2 = \frac{1}{\alpha'} \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

Open strings with DD boundary conditions

$$\dot{X}^\mu(\tau, \sigma) \Big|_{\sigma=0, \pi} = 0$$



$$\dot{X}^\mu(\tau, \sigma) = \frac{l_s^2}{2} (p^\mu + \bar{p}^\mu) - \frac{l_s}{\sqrt{2}} \sum_{k \neq 0} \left[\alpha_k^\mu e^{-ik(\tau-\sigma)} + \bar{\alpha}_k^\mu e^{-ik(\tau+\sigma)} \right]$$



$$\dot{X}^\mu(\tau, \sigma = 0) = \frac{l_s^2}{2} (p^\mu + \bar{p}^\mu) + \frac{l_s}{\sqrt{2}} \sum_{k \neq 0} e^{-ik\tau} (\alpha_k^\mu + \bar{\alpha}_k^\mu)$$

Then

$$p^\mu = -\bar{p}^\mu, \quad \alpha_k^\mu = -\bar{\alpha}_k^\mu$$

$$\dot{X}^\mu(\tau, \sigma) = -i\sqrt{2}l_s \sum_{k \neq 0} \bar{\alpha}_k^\mu \sin k\sigma e^{-ik\tau}$$

$$\dot{X}^\mu(\tau, \sigma) \Big|_{\pi} = 0 \text{ implies } k \in \mathbb{Z}$$

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 \bar{p}^\mu \sigma + \sqrt{2} l_s \sum_{n \in \mathbb{Z} - \{0\}} \frac{\bar{\alpha}_n^\mu}{n} e^{-in\tau} \sin n\sigma$$

Change

$$\bar{\alpha}_n^\mu \rightarrow \alpha_n^\mu$$

As

$$X_0^\mu = x^\mu ,$$

$$X_\pi^\mu = x^\mu + \pi l_s^2 \bar{p}^\mu$$



$$X^\mu(\tau, \sigma) = X_0^\mu + \frac{X_\pi^\mu - X_0^\mu}{\pi} \sigma + \sqrt{2} l_s \sum_{n \in \mathbb{Z} - \{0\}} \frac{\alpha_n^\mu}{n} e^{-in\tau} \sin n\sigma$$