String Theory

Lecture 3

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

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Quantization of the string

Substitute Poisson brackets by commutators according to the rule :

$$[\cdots]_{PB} \longrightarrow i\,[\cdots]$$

$$[A, B]_{PB} = C \qquad\qquad i\,[\hat{A}, \hat{B}] = \hat{C} \implies [\hat{A}, \hat{B}] = -i\hat{C}$$

Canonical commutation relations

$$\big[\,X^\mu(\tau,\sigma)\,,\,X^\nu(\tau,\sigma')\,\big] \;=\; 0$$

$$\big[\,\Pi^\mu(\tau,\sigma)\,,\,\Pi^\nu(\tau,\sigma')\,\big] \;=\; 0$$

$$\big[\,X^\mu(\tau,\sigma)\,,\,\Pi^\nu(\tau,\sigma')\,\big] \;=\; i\eta^{\mu\nu}\delta(\sigma-\sigma')$$

In terms of modes:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}$$

$$[\alpha^\mu_m\,,\,\alpha^\nu_n] \,=\, [\bar\alpha^\mu_m\,,\,\bar\alpha^\nu_n] \,=\, m\,\eta^{\mu\nu}\delta_{m+n}$$

$$[\alpha^\mu_m\,,\,\bar\alpha^\nu_n] \,=\, 0$$

Define

$$a^\mu_m \equiv \frac{1}{\sqrt{m}}\,\alpha^\mu_m\,, \qquad a^{\mu\dagger}_m \equiv \frac{1}{\sqrt{m}}\,\alpha^\mu_{-m}\,, \qquad m > 0$$

$$\bar a^\mu_m \equiv \frac{1}{\sqrt{m}}\,\bar\alpha^\mu_m\,, \qquad \bar a^{\mu\dagger}_m \equiv \frac{1}{\sqrt{m}}\,\bar\alpha^\mu_{-m}\,, \qquad m > 0$$

They satisfy

$$[\,a^\mu_m, a^{\nu\dagger}_n\,] \,=\, [\,\bar a^\mu_m, \bar a^{\nu\dagger}_n\,] \,=\, \eta^{\mu\nu}\,\delta_{m,n}$$

algebra of creation and annihilation operators of an infinite set of oscillators

Hilbert space

Ground state $|0;k>$

$$\alpha^\mu_m\,|0;k>=\,\bar{\alpha}^\mu_m\,|0;k>=\,0\,,\qquad\qquad m>0$$

The state $|0;k>$ is an eigenstate of p^μ:

$$p^\mu\,|0;k>=\,k^\mu\,|0;k>$$

General state

$$\alpha^{\mu_1}_{-m_1}\alpha^{\mu_2}_{-m_2}\cdots\bar{\alpha}^{\nu_1}_{-n_1}\bar{\alpha}^{\nu_2}_{-n_2}\cdots|0;k>$$

-are one-particle states

-carry different representation of the Lorentz group

Notice that the Hilbert space is not positive definite:

$$\eta^{00} = -1 \quad \Longrightarrow \quad [a^0_m, a^{0\dagger}_m] = -1$$

$$|| \, |a^{0\dagger}_m |0> ||^2 = < 0| \, a^0_m a^{0\dagger}_m \, |0> = < 0| \, [a^0_m, a^{0\dagger}_m] \, |0> = -1$$

We need subsidiary conditions to remove negative norm states

provided by the Virasoro constraints $L_m = \bar{L}_m = 0$

It is not possible to require these constraints as operator equations due to the (quantum)Virasoro algebra

$$[\, L_m \, , \, L_n \,] = (n - m) \, L_{n+m} + \frac{c}{12} \, m \, (m^2 - 1) \, \delta_{m+n}$$

$c = D$ is a quantum anomaly called the central charge

We require:

$$< \psi | \, L_m \, | \psi > = < \psi | \, \bar{L}_m \, | \psi > = 0$$

$$| \psi > \rightarrow \text{physical state}$$

Virasoro constraints in a weak sense

It is enough to impose

$$L_m \, | \psi > = \bar{L}_m \, | \psi > = 0 \, , \qquad m \geq 0$$

Since

$$< \psi | \, L_n^\dagger = 0 \, , \qquad n > 0 \qquad\qquad L_n^\dagger = L_{-n}$$

$$< \psi | \, L_{-n} \, | \psi > = < \psi | \, L_n^\dagger \, | \psi > = < \psi | \, L_n \, | \psi >^* = 0$$

Normal ordering:

annihilation operators are at the right of the creation operators

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{n=+\infty} : \alpha_{m-n}\, \alpha_n :$$

ambiguity in L_0 and $\bar{L}_0$ ➡ a c-number constant is generated in the reordering

we take

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n \ , \qquad \bar{L}_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{+\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n$$

and write the physical state condition as:

$$(L_m - a\delta_m)\,|\psi> = (\bar{L}_m - a\delta_m)\,|\psi> = 0 \ , \qquad m \geq 0$$

$$a \longrightarrow \text{c-number to be determined (zero-point energy)}$$

Non-covariant approach

Fix completely the gauge symmetry at the expense of explicit Lorentz covariance

Remaining symmetry

$$\xi^{\pm} \longrightarrow F_{\pm}(\xi^{\pm})$$

$$\tau \rightarrow \tilde{\tau} = \frac{1}{2}\left[F_{+}(\xi^{+}) + F_{-}(\xi^{-}) \right] , \qquad \sigma \rightarrow \tilde{\sigma} = \frac{1}{2}\left[F_{+}(\xi^{+}) - F_{-}(\xi^{-}) \right]$$

$\tilde{\tau}$ arbitrary solution of $\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) \tilde{\tau} = 0$

$\tilde{\tau}$ can be taken to be a combination of the X^{μ}

Light-cone coordinates in spacetime

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right) \qquad X^\mu = (X^+, X^-, X^i)\ , \qquad i = 1, \cdots D-2$$

$$V \cdot W = -V^+ W^- - V^- W^+ + \sum_i V^i W^i$$

$$V^- = -V_+\ , \qquad V^+ = -V_-\ , \qquad V^i = V_i$$

$$V \cdot W = V^+ W_+ + V^- W_- + \sum_i V^i W^i$$

Light-cone gauge

$$\tilde{\tau} \sim X^+ \qquad \tilde{\tau} = \frac{X^+}{l_s^2\, p^+} + \text{constant}$$

$$X^+ = x^+ + l_s^2\, p^+ \tau\ , \qquad \text{(closed strings)} \quad \Longrightarrow \quad \alpha_n^+ = 0$$

For the closed string

$$\left(\dot{X} \pm X'\right)^2 = 0$$

$$\dot{X}^+ = l_s^2\, p^+ \,, \qquad X^{+\prime} = 0$$

$$\Longrightarrow \quad \dot{X}^- \pm X^{-\prime} = \frac{1}{2 l_s^2\, p^+} \left(\dot{X}^i \pm X^{i\prime}\right)^2$$

Equivalently

$$\partial_\pm X^- = \frac{1}{l_s^2\, p^+} \left(\partial_\pm X^i\right)^2 \quad \Longrightarrow \quad \alpha_n^- = \frac{1}{\sqrt{2}\, l_s\, p^+} \sum_{n\in\mathbb{Z}} \alpha^i_{n-m}\, \alpha^i_m$$

Taking in account the normal ordering

$$\alpha_n^- = \frac{1}{\sqrt{2}\, l_s\, p^+} \left[\sum_{n\in\mathbb{Z}} :\alpha^i_{n-m}\, \alpha^i_m: - 2a\delta_n\right]$$

$$\bar{\alpha}_n^- = \frac{1}{\sqrt{2}\, l_s\, p^+} \left[\sum_{n\in\mathbb{Z}} :\bar{\alpha}^i_{n-m}\, \bar{\alpha}^i_m: - 2a\delta_n\right]$$

For the open string

$$X^+ = x^+ + 2 l_s^2\, p^+\, \tau$$

$$\partial_\pm X^- = \frac{1}{2 l_s^2\, p^+} \left(\partial_\pm X^i\right)^2 \quad \Longrightarrow \quad \alpha_n^- = \frac{1}{2\sqrt{2}\, l_s\, p^+} \left[\sum_{n\in\mathbb{Z}} :\alpha^i_{n-m}\, \alpha^i_m: - 2a\delta_n\right]$$

Mass spectrum of the open string

Define the number operator as:

$$N \equiv \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \alpha^i_{-n}\, \alpha^i_n \;=\; \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} n\, a^{i\,\dagger}_n\, a^i_n$$

$$L_0 \;=\; -\alpha'\, M^2 \;+\; N$$

$$(L_0 - a)\,|\,\psi >=\; 0$$

$$M^2 \;=\; \frac{1}{\alpha'}\,(\,N - a\,)$$

N is the eigenvalue of the number operator

Open string states

$N = 0$ → vacuum $|0>$

$$M^2 = -\frac{a}{\alpha'}$$ → tachyonic if $a > 0$

$N = 1$ → first excited state $\alpha^i_{-1}\,|0>$

$$M^2 = \frac{1-a}{\alpha'}$$

vector in $SO(D-2)$ with $D-2$ components

Determination of the normal ordering constant

Let us reorder L_0

$$L_0 = -\alpha' M^2 + \frac{1}{2} \sum_{i=1}^{D-2} \sum_{n=1}^{+\infty} \left(\alpha^i_{-n} \alpha^i_n + \alpha^i_n \alpha^i_{-n} \right)$$

As:

$$\alpha^i_n \alpha^i_{-n} = \alpha^i_{-n} \alpha^i_n + n$$

$$\Downarrow$$

$$L_0 = -\alpha' M^2 + \sum_{i=1}^{D-2} \sum_{n=1}^{+\infty} : \alpha^i_{-n} \alpha^i_n : + \frac{D-2}{2} \sum_{n=1}^{+\infty} n$$

a is the infinite constant

$$a = -\frac{D-2}{2} \sum_{n=1}^{+\infty} n$$

Regularize the infinite sum as:

$$\sum_{n=1}^{+\infty} n \implies \sum_{n=1}^{+\infty} n\, e^{-\epsilon\, n}\,, \qquad \epsilon > 0,\ \ \epsilon \to 0$$

As

$$\sum_{n=1}^{+\infty} n\, e^{-\epsilon\, n} = -\frac{d}{d\epsilon} \sum_{n=0}^{+\infty} e^{-\epsilon\, n} = -\frac{d}{d\epsilon}\left[\frac{1}{1-e^{-\epsilon}}\right] =$$

$$= -\frac{d}{d\epsilon}\left[\frac{1}{\epsilon}\left[1-\frac{\epsilon}{2}+\frac{\epsilon^2}{6}+\cdots\right]^{-1}\right]$$

one has:

$$\sum_{n=1}^{+\infty} n\, e^{-\epsilon\, n} = \frac{1}{\epsilon^2} - \frac{1}{12} + o(\epsilon)$$

Thus:

$$\boxed{\sum_{n=1}^{+\infty} n \to -\frac{1}{12}} \quad \Longrightarrow \quad \boxed{a = \frac{D-2}{24}}$$

Alternative determination of a

Define the Riemann ζ-function as:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} , \qquad z \in \mathbb{C} , \; Re(z) > 1$$

can be analytically continued to the whole complex plane

It satisfies the relation

$$\pi^z \zeta(1-z) = 2^{1-z} \Gamma(z) \cos \frac{\pi z}{2} \zeta(z)$$

The regulated version of $\sum_{n=1}^{+\infty} n$ should be $\zeta(-1)$

$$\zeta(-1) = -\frac{1}{2\pi^2} \zeta(2)$$

↓

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \Rightarrow \quad \boxed{\zeta(-1) = -\frac{1}{12}}$$

same value as before

Open string spectrum

A tachyon scalar $|0>$ with mass

$$M^2 = -\frac{D-2}{24\,\alpha'}$$

A vector $\alpha^i_{-1}\,|0>$ with mass

$$M^2 = \frac{1}{\alpha'}\left(1 - \frac{D-2}{24}\right) = \frac{26-D}{24\,\alpha'}$$

This vector has D-2 components and in a relativistic theory should be massless

Fixes the dimension of the target space to be:

$$\boxed{D = 26}$$

Critical dimension of the bosonic string

D=26 → $a = 1$

$$M^2(tachyon) = -\frac{1}{\alpha'}$$

Excited states

The states with $N = 2$ are: $\alpha^i_{-2}\,|0>$, $\alpha^i_{-1}\,\alpha^j_{-1}\,|0>$

Their mass is $M^2 = \frac{1}{\alpha'}$

number of states in D dimensions:

$$D - 2 + \frac{1}{2}\,(D-2)(D-1) = \frac{D(D-1)}{2} - 1$$

massive particle of spin 2

In general

$$J \leq \alpha'\, M^2 + 1$$

Regge trajectories

The spectrum is a tower of states with increasing mass and spin

massive states are infinitely massive when $\alpha' \to 0$

Lorentz symmetry

Generator of Lorentz transformations

$$J^{\mu\nu} = \int d\sigma \, J^{\mu\nu}_{\tau} = T \int d\sigma \left[X^{\mu} \dot{X}^{\nu} - X^{\nu} \dot{X}^{\mu} \right]$$

In terms of modes

Closed strings ⟶ $J^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu} + E^{\mu\nu} + \bar{E}^{\mu\nu}$

Open strings ⟶ $J^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu} + E^{\mu\nu}$

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^{\mu}_{-n} \alpha^{\nu}_{n} - \alpha^{\nu}_{-n} \alpha^{\mu}_{n} \right)$$

$$\bar{E}^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\bar{\alpha}^{\mu}_{-n} \bar{\alpha}^{\nu}_{n} - \bar{\alpha}^{\nu}_{-n} \bar{\alpha}^{\mu}_{n} \right)$$

Poincare algebra

$$[\,p^\mu, p^\nu\,] = 0$$

$$[\,p^\mu, J^{\nu\rho}\,] = -i\eta^{\mu\nu}\, p^\rho + i\eta^{\mu\rho}\, p^\nu$$

$$[\,J^{\mu\nu}, J^{\rho\lambda}\,] = -i\eta^{\nu\rho}\, J^{\mu\lambda} + i\eta^{\mu\rho}\, J^{\nu\lambda} + i\eta^{\nu\lambda}\, J^{\mu\rho} - i\eta^{\mu\lambda}\, J^{\nu\rho}$$

In a covariant gauge this algebra is satisfied for any D and a

But unitarity (absence of negative norm states) only when:

D=26 and a=1 → No ghost theorem

In the light-cone gauge

Poincare algebra non-trivial because $\alpha^- \sim \sum_i \alpha^i \alpha^i$

$\left[J^{i-}, J^{j-} \right]$ should be zero

One gets:

$$\left[J^{i-}, J^{j-} \right] \sim \sum_{m=1}^{\infty} \Delta_m \left[\alpha^i_{-m} \alpha^j_m - \alpha^j_{-m} \alpha^i_m \right]$$

$$\Delta_m = m \left[\frac{26 - D}{12} \right] + \frac{1}{m} \left[\frac{D - 26}{12} + 2(1 - a) \right]$$

Δ_m should vanish for any m

D=26 , a=1

Only for these values Lorentz invariance is unbroken