String Theory

Lecture 4

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

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Covariant quantization of the open string

We study the representation theory of the Virasoro algebra

$$[\,L_m\,,\,L_n\,] \,=\, (n-m)\,L_{n+m} \,+\, \frac{c}{12}\,m\,(m^2-1)\,\delta_{m+n} \qquad c = D$$

graded by $N \,=\, \sum_{n=1}^{+\infty} \alpha^{\mu}_{-n}\,\alpha_{n\,\mu}$ $\qquad [N\,,\,\alpha^{\mu}_{-n}] \,=\, n\,\alpha^{\mu}_{-n}\,, \qquad n > 0$

Virasoro primaries

$$(L_0 - a)\,|\phi>=\,0$$

$$L_n\,|\phi>=\,0\,, \qquad n > 0$$

The number a is called the weight of the Virasoro primary

Virasoro descendants of a primary

States that can be written as a finite linear combination of products of Virasoro operators with negative modes acting on the primary

First descendants

$$L_{-1}\,|\phi> \longrightarrow \text{level one}$$

$$L_{-2}\,|\phi>\ ,\ L_{-1}\,L_{-1}\,|\phi> \longrightarrow \text{level two}$$

$$L_{-3}\,|\phi>\ ,\ L_{-2}\,L_{-1}\,|\phi>\ ,L_{-1}\,L_{-2}\,|\phi>\ ,\ L^2_{-1}\,|\phi> \longrightarrow \text{level three}$$

They have the structure of a Verma module

They are not all independent since

$$L_{-1}\,L_{-2} = [L_{-1}\,,\,L_{-2}] + L_{-2}\,L_{-1} = L_{-3} + L_{-2}\,L_{-1}$$

Basis at level $N_\phi + n$:

$$L_{-n_1} L_{-n_2} \cdots L_{-n_k} |\phi > \qquad n = \sum_i n_i$$

$$n_1 \geq n_2 \cdots \geq n_k$$

The descendants are orthogonal to any primary

Take a general descendant of the form

$$|des >= \sum_i c_i L_{-n_i} |\chi_i >$$

$$c_i \rightarrow \text{constants}, \ |\chi_i > \rightarrow \text{states}, \ n_i > 0$$

$$\Downarrow$$

$$< des|primary >= \sum_i c_i^* < \chi_i | L_{n_i} | primary >= 0$$

A null state is a state which is both primary and descendant

$|\phi >$ and $|\phi > + |null > \rightarrow$ the same inner products with all primary states

A null state has zero norm and the primary states that differ by a null state are physically indistinguishable

⇓

$$|\phi > \rightarrow |\phi > + |null > \text{ is a (gauge) symmetry}$$

A physical state is an equivalence class of a primary state of weight a=1 modulo the null states

$$|phys > \sim |phys > + |null >$$

Vector states of the open string in covariant form

$$\xi_\mu\, \alpha^\mu_{-1}\, |0; k> \equiv |\xi; k>$$

$\xi_\mu \rightarrow$ polarization vector

$(L_0 - 1)\, |\xi; k> = 0$ implies that $M = 0$

Physical state condition

$$L_1\, |\xi; k> = (\alpha_0 \cdot \alpha_1 + \alpha_{-1} \cdot \alpha_2 \cdots)\, \xi_\mu\, \alpha^\mu_{-1}\, |0; k> =$$

$$= \alpha_{0\,\nu}\, [\alpha^\nu_1, \alpha^\mu_{-1}]\, \xi_\mu\, |0; k> = \xi_\mu\, \alpha^\mu_0\, |0; k>$$

$$\Downarrow$$

$$L_1\, |\xi; k> = \sqrt{2}\, l_s\, (\xi \cdot k)\, |0; k>$$

$$\Downarrow$$

$$L_1\, |\xi; k> = 0 \Longrightarrow \xi \cdot k = 0$$

Transversality condition : polarization and momentum are orthogonal

Let us consider the descendant state

$$|d> = \frac{\lambda}{\sqrt{2}\, l_s} L_{-1}\, |0;k>$$

$$L_{-1}\, |0;k> = \left(\alpha_{-1} \cdot \alpha_0 + \alpha_{-2} \cdot \alpha_1 + \cdots \right) |0;k> = \alpha_{-1} \cdot \alpha_0\, |0;k> = \sqrt{2}\, l_s\, k_\mu\, \alpha^\mu_{-1}\, |0;k>$$

$$|d> = \lambda\, k_\mu\, \alpha^\mu_{-1}\, |0;k>$$

⇒ $|d>$ is a vector state with polarization $\hat{\xi}^\mu = \lambda\, k^\mu$

This state is physical if $k^2 = 0$ because $k_\mu \hat{\xi}^\mu = \lambda\, k^2 = 0$

If $k^2 = 0$ the state $|d>$ is physical and descendant and thus null $\quad < d|d> = |\lambda|^2\, k^2 = 0$

⇓

ξ_μ and $\xi_\mu + \lambda k^\mu$ are equivalent

⇓

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

gauge symmetry!

Closed string spectrum

Physical conditions $\Rightarrow$

$$(L_0 - 1)\,|\psi> = 0\,, \qquad (\bar{L}_0 - 1)\,|\psi> = 0$$

$$(L_0 - \bar{L}_0)\,|\psi> = 0$$

In the light-cone gauge $\alpha_n^+ = \bar{\alpha}_n^+ = 0$

$$L_0 = -\frac{\alpha'}{4} M^2 + N\,, \qquad \bar{L}_0 = -\frac{\alpha'}{4} M^2 + \bar{N}$$

with

$$N \equiv \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \alpha_{-n}^i\, \alpha_n^i \qquad \bar{N} \equiv \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \bar{\alpha}_{-n}^i\, \bar{\alpha}_n^i$$

Thus, the mass formula for the closed string is:

$$(L_0 + \bar{L}_0 - 2)\,|\psi> = 0 \quad \Rightarrow \quad M^2 = \frac{2}{\alpha'}(N + \bar{N} - 2)$$

level-matching condition $N = \bar{N}$

First mass levels

$$N = \bar{N} = 0 \;\rightarrow\; |0> \;\rightarrow\; M^2 = -\frac{4}{\alpha'}$$

Closed string tachyon

$$N = \bar{N} = 1 \;\rightarrow\; \Omega^{ij} = \alpha^i_{-1}\,\bar{\alpha}^j_{-1}\,|0>$$

$$\downarrow$$

$$M^2 = 0$$

Massless states

Plus an infinite tower of massive states

We decompose Ω^{ij} in three parts

$$\Omega^{ij} = h^{ij} + B^{ij} + \frac{\Phi}{24}\delta^{ij}$$

$h^{ij} \Longrightarrow$ symmetric traceless

$B^{ij} \Longrightarrow$ antisymmetric

$\Phi \Longrightarrow$ trace

Inverse relations

$\Phi = \delta^{ij}\,\Omega^{ij}$ → Φ is the trace of Ω^{ij}

Φ is a scalar field that is called the dilaton

$h^{ij} = \frac{1}{2}\left[\Omega^{ij} + \Omega^{ji}\right] - \frac{\Phi}{24}\delta^{ij}$ → $h^{ij} = h^{ji}$ $\quad \delta^{ij}\,h^{ij} = 0$

h^{ij} is a spin two massless particle → **The graviton!**

$g^{\mu\nu} \approx \eta^{\mu\nu} + h^{\mu\nu}$ → **closed string theory contains gravity!**

$$B^{ij} = \frac{1}{2}\left[\Omega^{ij} - \Omega^{ji}\right] \Rightarrow \quad B^{ij} \text{ is a two-form gauge potential}$$

In a covariant form the massless fields of the closed string are:

$$(g_{\mu\nu}, B_{\mu\nu}, \Phi)$$

The null vectors of the Virasoro algebra generate general coordinate transformations and the gauge symmetry of the antisymmetric field

Closed string coupled to gravity

$$S[X] = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, G_{\mu\nu}(X)$$

Action of a 2d interacting field $X^\mu(\xi)$

Quantization with the background field method

Expand $X^\mu(\xi)$ around a classical configuration $X_0(\xi)$

$$X^\mu(\xi) = X_0^\mu(\xi) + \pi^\mu(\xi) \quad \Rightarrow \quad \pi^\mu \text{ is not a vector}$$

Riemann normal coordinates

$\lambda^\mu(t) \Rightarrow$ geodesic that connects X_0^μ and $X_0^\mu + \pi^\mu$

$$\ddot{\lambda}^\mu(t) + \Gamma^\mu_{\nu\sigma}\, \dot{\lambda}^\nu(t)\, \dot{\lambda}^\sigma(t) = 0$$

$$\lambda(0) = X_0^\mu$$

$$\lambda(1) = X_0^\mu + \pi^\mu$$

$$t \in [0,1]$$

Take $$\dot{\lambda}^\nu(0) = \sqrt{\alpha'}\,\eta^\mu$$

Solve iteratively

$$X^\mu(t) = X_0^\mu + \sqrt{\alpha'}\,\eta^\mu\,t - \frac{\alpha'}{2}\Gamma^\mu_{\sigma_1\sigma_2}\,\eta^{\sigma_1}\,\eta^{\sigma_2}\,t^2 - \frac{(\alpha')^{\frac{3}{2}}}{3}\,\Gamma^\mu_{\sigma_1\sigma_2\sigma_3}\,\eta^{\sigma_1}\,\eta^{\sigma_2}\,\eta^{\sigma_3}\,t^3 + \cdots$$

$$\Gamma^\mu_{\sigma_1\sigma_2\sigma_3} = \nabla_{\sigma_1}\Gamma^\mu_{\sigma_1\sigma_3}$$

⇓

$$\pi^\mu = \sqrt{\alpha'}\,\eta^\mu - \frac{\alpha'}{2}\Gamma^\mu_{\sigma_1\sigma_2}\,\eta^{\sigma_1}\,\eta^{\sigma_2} + \cdots$$

$\eta^\mu(\xi)$ parametrizes the quantum fluctuation

α' is an expansion parameter

Expansion of the action

$$S[X] = S[X_0] + S^{(2)}[X_0, \pi] + \cdots$$

Expansion in powers of α' ⇒ Low energy expansion

$$S[X_0] = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X_0^\mu\, \partial_\beta X_0^\nu\, G_{\mu\nu}(X_0)$$

$$S^{(2)}[X_0,\pi] = -\frac{1}{4\pi} \int d^2\xi \sqrt{g}\, g^{\alpha\beta} \left[G_{\mu\nu}(X_0)\partial_\alpha \eta^\mu\, \partial_\beta \eta^\nu + R_{\mu\lambda\sigma\nu}(X_0)\, \partial_\alpha X_0^\mu\, \partial_\beta X_0^\nu\, \eta^\lambda \eta^\sigma \right]$$

Quantum correction

$$-\frac{1}{4\pi} R_{\mu\lambda\sigma\nu}(X_0)\, \partial_\alpha X_0^\mu\, \partial_\beta X_0^\nu\, < \eta^\lambda \eta^\sigma > \quad \Rightarrow \quad \text{divergent}$$

In dimensional regularization in $2+\epsilon$ dimensions:

$$< \eta^\lambda \eta^\sigma > = 2\pi\delta^{\lambda\sigma}\, \mu^{-\epsilon} \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{i}{k^2 - m^2 + i0}$$

μ is the renormalization point

m is an IR mass scale

Performing the integral

$$< \eta^\lambda \, \eta^\sigma > = \frac{\delta^{\lambda\sigma}}{2} \left(\frac{m}{\sqrt{4\pi}\,\mu} \right)^{\epsilon} \Gamma\left(-\frac{\epsilon}{2} \right) \; = \; -\frac{1}{\epsilon}\,\delta^{\lambda\sigma} \; + \text{regular terms}$$

It induces the divergent term $\Rightarrow$ $\frac{1}{4\pi\epsilon} \int d^2\xi \sqrt{g}\; g^{\alpha\beta}\, \partial_\alpha X_0^\mu \, \partial_\beta X_0^\nu \, R_{\mu\nu}(X_0)$

$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ is the Ricci tensor

It can be reabsorbed with the following renormalization of the metric

$$\boxed{G_{\mu\nu} \rightarrow G_{\mu\nu} + \frac{\alpha'}{\epsilon} R_{\mu\nu}}$$

which corresponds to the beta function

$$\boxed{\beta_{\mu\nu} \; = \; \mu \, \frac{\partial G_{\mu\nu}}{\partial \mu} \; = \; \alpha' \, R_{\mu\nu}}$$

Ricci flow

A non-zero beta function signals breaking of 2d scale invariance

$$\xi^\alpha \to \lambda \xi^\alpha$$

Scale transformations induce Weyl transformations

$$ds^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta \to \lambda^2 g_{\alpha\beta} d\xi^\alpha d\xi^\beta \Longrightarrow g_{\alpha\beta} \to \lambda^2 g_{\alpha\beta}$$

If Weyl symmetry is broken we get an inconsistency

↓

We should require

$$\boxed{\beta_{\mu\nu} = 0 \Longrightarrow R_{\mu\nu} = 0}$$

classical Einstein equations are obtained as a quantum effect in 2d!

Let us fix the conformal gauge in the quantum theory

$$g_{\alpha\beta} = e^{2\phi}\,\delta_{\alpha\beta}$$

Action in $2+\epsilon$ dimensions:

$$S = -\frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\xi\, e^{\epsilon\phi}\,\partial_\alpha X^\mu\,\partial^\alpha X^\nu\, G_{\mu\nu}(X)$$

The conformal factor ϕ does not decouple in $2+\epsilon$ dimensions

$$e^{\epsilon\phi} \approx 1+\epsilon\phi \qquad\qquad S \approx -\frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\xi\,(1+\epsilon\phi)\,\partial_\alpha X^\mu\,\partial^\alpha X^\nu\, G_{\mu\nu}(X)$$

Renormalizing the metric we get an extra finite term

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\,\partial_\alpha X^\mu\,\partial^\alpha X^\nu\Big(G_{\mu\nu}(X) + \alpha'\,\phi\, R_{\mu\nu}(X)\Big)$$

The action depends on the conformal factor

$$\frac{\delta S}{\delta \phi} = -\frac{1}{4\pi}\, \partial_\alpha X^\mu \, \partial^\alpha X^\nu \, R_{\mu\nu}(X)$$

Weyl transformation

$$g_{\alpha\beta} \longrightarrow e^{2\delta\phi}\, g_{\alpha\beta} \quad \Rightarrow \quad \delta\, g_{\alpha\beta} = 2\delta\phi\, g_{\alpha\beta}\,, \quad \delta\, g^{\alpha\beta} = -2\delta\phi\, g^{\alpha\beta}$$

$$\delta S = \frac{\delta S}{\delta g^{\alpha\beta}}\, \delta g^{\alpha\beta} = T\sqrt{-\det g}\;\, T^\alpha_\alpha\, \delta\phi$$

For a flat worldsheet metric $\Rightarrow$

$$\frac{\delta S}{\delta \phi} = \frac{1}{2\pi\alpha'}\, T^\alpha_\alpha$$

$\Downarrow$

$$T^\alpha{}_\alpha = -\frac{\alpha'}{2}\, R_{\mu\nu}(X)\, \partial_\alpha X^\mu\, \partial^\alpha X^\nu = -\frac{\beta_{\mu\nu}}{2}\, \partial_\alpha X^\mu\, \partial^\alpha X^\nu$$

Weyl invariance (and thus consistency) only if $\beta_{\mu\nu} = 0$

Coupling to the antisymmetric $B_{\mu\nu}$ field

$$S' = -\frac{1}{4\pi\alpha'} \int d^2\xi \; B_{\mu\nu}(X)\, \partial_\alpha X^\mu \, \partial_\beta X^\nu \, \epsilon^{\alpha\beta}$$

$\epsilon^{\alpha\beta}$ is an antisymmetric tensor with $\epsilon^{01} = +1$

The action changes by a total derivative under

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \quad \Longrightarrow \quad \text{gauge symmetry}$$

Coupling to the dilaton

$$S'' = \frac{1}{4\pi} \int d^2\xi \; \sqrt{g}\, R^{(2)}\, \Phi(X)$$

Total Weyl anomaly

$$T^{\alpha}{}_{\alpha} = -\frac{1}{2}\,\beta_{\mu\nu}(G)\,g^{\alpha\beta}\,\partial_\alpha\,X^\mu\,\partial_\beta\,X^\nu - \frac{1}{2}\,\beta_{\mu\nu}(B)\,\epsilon^{\alpha\beta}\,\partial_\alpha\,X^\mu\,\partial_\beta\,X^\nu - \frac{\alpha'}{2}\,\beta(\Phi)\,R^{(2)}$$

$$\beta_{\mu\nu}(G) \;=\; \alpha'\,R_{\mu\nu} \;+\; 2\alpha'\,\nabla_\mu\,\nabla_\nu\,\Phi \;-\; \frac{\alpha'}{4}\,H_{\mu\lambda\sigma}\,H_\nu{}^{\lambda\sigma}$$

$$\beta_{\mu\nu}(B) \;=\; -\frac{\alpha'}{2}\,\nabla^\lambda\,H_{\lambda\mu\nu} \;+\; \alpha'\,\nabla^\lambda\Phi\,H_{\lambda\mu\nu}$$

$$\beta_{\mu\nu}(\Phi) \;=\; -\frac{\alpha'}{2}\,\nabla^2\,\Phi \;+\; \alpha'\,\nabla_\mu\Phi\nabla^\mu\Phi \;-\; \frac{\alpha'}{24}\,H_{\mu\nu\lambda}\,H^{\mu\nu\lambda}$$

$H_{\mu\nu\rho}$ is the field strength of $B_{\mu\nu}$

$$H_{\mu\nu\rho} \;=\; \partial_\mu\,B_{\nu\rho} \;+\; \partial_\nu\,B_{\rho\mu} \;+\; \partial_\rho\,B_{\mu\nu}$$

$H_{\mu\nu\rho}$ is invariant under the gauge transformation of $B_{\mu\nu}$

EOMs of the massless fields

$$\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta_{\mu\nu}(\Phi) = 0$$

They can be derived from the low energy effective action

$$S = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-G}\, e^{-2\Phi} \left[R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi\, \partial^\mu \Phi \right]$$

$R = R^i{}_i$ is the scalar curvature $\qquad \kappa \sim l_s^{24}$

Change to Einstein frame

$$G^E_{\mu\nu} = e^{-\frac{\phi}{6}} G_{\mu\nu}$$

$$S = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-G^E} \left[R^E - \frac{1}{12} e^{-\frac{\Phi}{3}} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial_\mu \Phi\, \partial^\mu \Phi \right]$$

Next order in $\alpha' \Longrightarrow$ higher powers of the curvature R