String Theory

Lecture 5

Master en Física Nuclear e de Partículas e as súas aplicacións Tecnolóxicas e Médicas

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Consider a theory in d Euclidean dimensions

Infinitesimal change of coordinates ➡ $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$

The change of action is:

$$\delta S \sim \int d^d x \, T^{\mu\nu} \, \partial_\mu \, \epsilon_\nu$$

$T^{\mu\nu}$ is the energy momentum tensor $\quad T^{\mu\nu} = T^{\nu\mu} \quad \partial_\mu T^{\mu\nu} = 0$

Let us consider a scale transformation

$x^\mu \rightarrow \lambda x^\mu$ with $\lambda = 1 + \epsilon$ $\quad \epsilon$ small ➡ $\epsilon^\mu = \epsilon \, x^\mu$

➡ $\delta S \sim \int d^d x \, T^\mu{}_\mu$ ➡ $\delta S = 0 \Longrightarrow T^\mu{}_\mu = 0$

If the system is scale invariant the EM tensor is traceless

We have a Conformal Field Theory

2d theory in Euclidean space → $x^\mu = (x^0, x^1)$

complex variables → $z = x^0 + i x^1 \,, \qquad \bar{z} = x^0 - i x^1$

$$ds^2 = (dx^0)^2 + (dx^1)^2 = dz\, d\bar{z} \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0 \,, \qquad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$$

EM tensor

$$T^\mu{}_\mu = 4 T_{z\bar{z}}$$ → scale invariance → $T_{z\bar{z}} = 0$

Conformal (analytic) transformation → $z \to z + \epsilon(z) \,, \qquad \bar{z} \to \bar{z} + \overline{\epsilon(z)}$

$$\epsilon^z = \epsilon(z) \,, \qquad \epsilon^{\bar{z}} = \overline{\epsilon(z)} \,, \qquad \epsilon_z = \frac{1}{2}\overline{\epsilon(z)} \,, \qquad \epsilon_{\bar{z}} = \frac{1}{2}\epsilon(z)$$

↓

$$T^{\mu\nu}\, \partial_\mu\, \epsilon_\nu = T^{z\bar{z}} \left(\partial_z\, \epsilon_{\bar{z}} + \partial_{\bar{z}}\, \epsilon_z \right)$$

↓

$\delta S = 0$ for any conformal transformation → scale invariance is promoted to an infinite dimensional symmetry!

From the conservation of the EM tensor

$$\partial^{\mu} T_{\mu\nu} = 0$$

In complex variables

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}\,z} = 0 \qquad \partial_z T_{\bar{z}\,\bar{z}} + \partial_{\bar{z}} T_{z\,\bar{z}} = 0$$

In a CFT → $T_{\bar{z}\,z} = T_{z\,\bar{z}} = 0$

↓

$$\partial_{\bar{z}} T_{zz} = 0 \Longrightarrow T_{zz} \text{ is holomorphic} \Longrightarrow \boxed{T_{zz} = T_{zz}(z)}$$

$$\partial_z T_{\bar{z}\,\bar{z}} = 0 \Longrightarrow T_{\bar{z}\,\bar{z}} \text{ is antiholomorphic} \Longrightarrow \boxed{T_{\bar{z}\,\bar{z}} = T_{\bar{z}\bar{z}}(\bar{z})}$$

Primary fields

Under a confomal transformation ⇒ $z \to f(z)\ ,\ \bar{z} \to \overline{f}(\bar{z})$

They transform as:

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \overline{f}}{\partial \bar{z}}\right)^{\overline{h}} \phi(f(z),\overline{f}(\bar{z}))$$

$(\,h,\overline{h}\,) \to$ conformal weights of ϕ

$\Delta = h + \bar{h}$ is the scaling dimension of ϕ

Infinitesimal transformations

$$f(z) = z + \epsilon(z)\quad ,\quad \overline{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$$

⇓

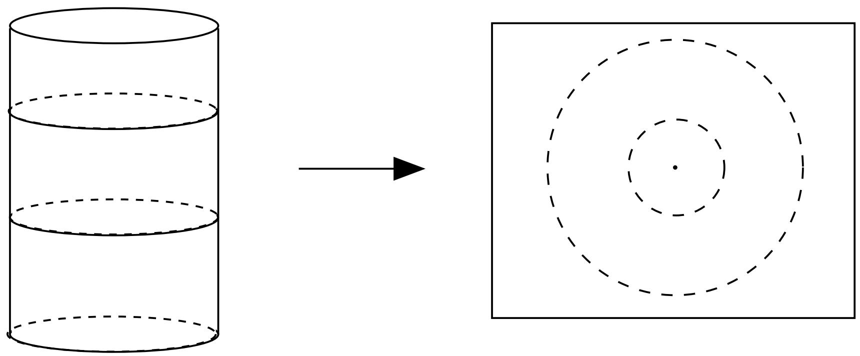
$$\delta_{\epsilon,\overline{\epsilon}}\phi(z,\overline{z}) = \left[(h\partial\epsilon + \epsilon\partial) + (\overline{h\partial\epsilon} + \overline{\epsilon\partial})\right]\phi(z,\overline{z})$$

Radial quantization

System with finite spatial dimension $0 \leq x^1 \leq L$, $x^1 \sim x^1 + L$ ➡ $\zeta = x^0 + ix^1 \quad , \quad \bar{\zeta} = x^0 - ix^1$

map to the full plane

$$\zeta \rightarrow z = e^{\frac{2\pi}{L}\zeta} \qquad z = exp\left[\frac{2\pi}{L}(x^0 + ix^1)\right]$$



time flows in the radial direction

$$x^0 = const. \rightarrow |z| = const.$$

$$x^1 = const. \rightarrow arg(z) = const.$$

$$z = 0 \rightarrow \text{infinite past}$$

$$|z| = \infty \rightarrow \text{infinite future}$$

Time ordering goes to radial ordering defined as:

$$R[A(z)B(w)] = \begin{cases} A(z)B(w) \ if |z| > |w| \\ B(w)A(z) \ if |z| < |w| \end{cases}$$

equal-time commutators

$$Q(t) = \int dx \mathcal{O}(t,x) \Longrightarrow [Q(t), \phi(t,x')] = \int dx [\mathcal{O}(t,x), \phi(t,x')]$$

regulate by time splitting

$$[\mathcal{O}(t,x), \phi(t,x')] = \lim_{\epsilon \to 0^+} [\mathcal{O}(t+\epsilon, x)\phi(t,x') - \phi(t+\epsilon, x')\mathcal{O}(t,x)]$$

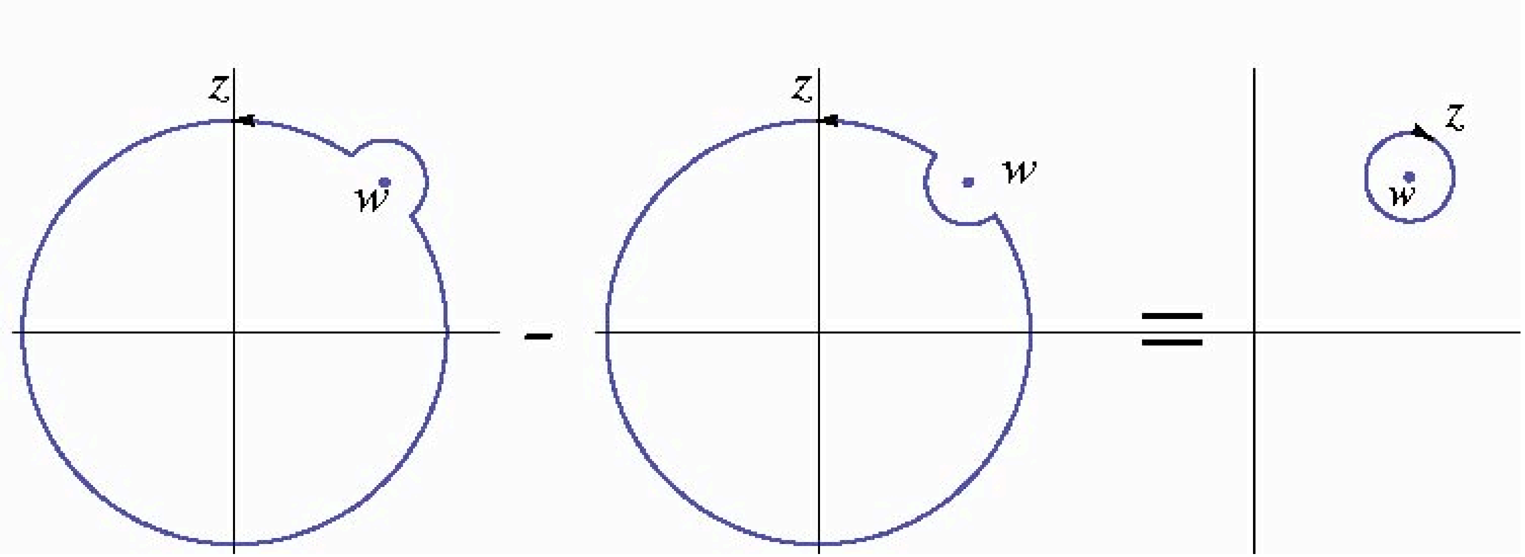
$$\Downarrow$$

$$[Q(t), \phi(t,x')] = \lim_{t \to t'} \left[\left[\int_{t>t'} dx - \int_{t<t'} dx \right] T\left[\mathcal{O}(t,x), \phi(t',x')\right] \right]$$

In radial quantization in the complex plane

$$Q = \frac{1}{2\pi i} \oint dz \mathcal{O}(z) \Longrightarrow \text{along a contour with } |z| \text{ fixed}$$

$$\Longrightarrow \quad [Q, \phi(w)]_{ET} = \lim_{|z| \to |w|} \left[\left[\oint_{|z|>|w|} - \oint_{|w|>|z|} \right] R[\mathcal{O}(z)\phi(w)] \frac{dz}{2\pi i} \right]$$



$$\oint_{|z|>|w|} - \oint_{|w|>|z|} = \oint_w \quad \Longrightarrow \quad \text{small circle around } z = w$$

$$\left[\oint \frac{dz}{2\pi i}\mathcal{O}(z), \phi(w)\right]_{ET} = \oint_w \frac{dz}{2\pi i} R[\mathcal{O}(z)\phi(w)]$$

$\Rightarrow$ rhs determined by the singularities as $z \to w$

Particular case

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z) - \frac{1}{2\pi i}\oint d\overline{z}\overline{\epsilon}(\overline{z})\overline{T}(\overline{z})$$

Q_ϵ should be the generator of $z \to z + \epsilon$, $\overline{z} \to \overline{z} + \overline{\epsilon}$

Consider the Operator Product Expansion (OPE)

$$R(T(z)\phi(w)) = \frac{h}{(z-w)^2}\phi(w) + \frac{\partial\phi(w)}{z-w} + regular\ terms$$

ϕ primary of weight h

⇓

$$[Q_\epsilon, \phi(w)] = \oint_C \frac{dz}{2\pi i}\epsilon(z)R\left[T(z)\phi(w)\right] = \oint_C \frac{dz}{2\pi i}\epsilon(z)\left[\frac{h}{(z-w)^2} + \frac{1}{z-w}\partial_w\phi(w)\right]$$

Using

$$\oint_C \frac{\epsilon(z)}{(z-w)^2}\frac{dz}{2\pi i} = \partial_w\epsilon(w) \ , \ \oint_C \frac{\epsilon(z)}{(z-w)}\frac{dz}{2\pi i} = \epsilon(w)$$

we get

$$\oint_C \frac{dz}{2\pi i}\epsilon(z)R[T(z)\phi(w)] = \epsilon(w)\partial_w\phi(w) + h\partial_w\epsilon(w) = \delta_\epsilon\phi(w)$$

The OPE with T characterizes the primary field

2d scalar field

$$S = \frac{g}{2}\int d^2x \partial_\mu \phi \, \partial^\mu \phi$$

g constant

EOM ➡ $\partial_z \partial_{\overline{z}} \phi(z, \overline{z}) = 0$ ➡ $\phi(z, \overline{z}) = \phi_R(z) + \phi_L(\overline{z})$

Green's function

$$\nabla^2 K(\vec{x}, \vec{y}) = -\frac{1}{g}\delta(\vec{x} - \vec{y}) \quad \Rightarrow \quad \langle \phi(\vec{x})\phi(\vec{y})\rangle = K(\vec{x}, \vec{y})$$

$K(\vec{x}, \vec{y})$ can only depend on $|\vec{x} - \vec{y}|$ ➡ $\nabla^2 K(|\vec{x}|) = -\frac{1}{g}\delta(\vec{x})$

Integrate over a disk D_r of radius r centered at $\vec{x} = 0$

$$\int_{D_r} d^2x \, \nabla^2 K(x) = -\frac{1}{g}$$

Polar coordinates $(r, \theta) \rightarrow K = K(r)$

$$\int_{D_r} \nabla^2 K(x) d^2x = \oint_{\partial D} \vec{\nabla} K \cdot \vec{n}\, dl \quad \Rightarrow \quad \oint_{\partial D} \vec{\nabla} K \cdot \vec{n}\, dl = \int_0^{2\pi} \frac{dK}{dr}\, r\, d\theta = 2\pi\, r\, \frac{dK}{dr}$$

$$\Rightarrow \quad \frac{dK}{dr} = -\frac{1}{2\pi g}\,\frac{1}{r} \quad \Rightarrow \quad \boxed{K(r) = -\frac{1}{2\pi g} ln(r) + constant}$$

Thus ➡ $$\langle \phi(\vec{x}) \phi(\vec{y}) \rangle = -\frac{1}{4\pi g} ln\, (\, \vec{x} - \vec{y}\,)^2 + constant$$

OPE of scalar fields

$$\phi_R(z)\, \phi_R(w) \sim -\frac{1}{4\pi g}\, ln(z - w) \;+\; \text{regular terms}$$

$$\phi_L(\bar{z})\, \phi_L(\bar{w}) \sim -\frac{1}{4\pi g}\, ln(\bar{z} - \bar{w}) \;+\; \text{regular terms}$$

Classical EM tensor

$$T_{\mu\nu} = g[-\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{2}\eta_{\mu\nu}\,\partial_\lambda\phi\,\partial^\lambda\phi]$$

Define

$$T(z) = 2\pi T_{zz} \;=\; -2\pi g\,\partial\phi_R\,\partial\phi_R \qquad \overline{T}(\overline{z}) = 2\pi T_{\overline{z}\overline{z}} \;=\; -2\pi g\,\overline{\partial}\phi_L\overline{\partial}\phi_L$$

Quantum mechanically we have to normal order

$$T = -2\pi g : \partial\phi\partial\phi :\equiv -2\pi g \lim_{z\to w}[\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle]$$

$$\langle\partial\phi(z)\partial\phi(w)\rangle = \partial_z\partial_w\,\langle\phi(z)\phi(w)\rangle \;=-\frac{1}{4\pi g}\frac{1}{(z-w)^2}$$

⇓

$$T(z) = -2\pi g \lim_{z\to w}\left[\partial\phi(z)\partial\phi(w) + \frac{1}{4\pi g}\frac{1}{(z-w)^2}\right]$$

$$j(z) \equiv \partial\,\phi(z) \to \text{primary field of weight } h = 1$$

$$T(z)\partial\phi(w) = -2\pi g : \partial_z\phi\partial_z\phi : \partial_w\phi(w) \sim -4\pi g \left[-\frac{1}{4\pi g}\frac{1}{(z-w)^2} \right] \partial_z\phi(z) =$$

$$= \frac{1}{(z-w)^2}\left[\partial_w\phi(w) + (z-w)\partial^2\phi(w) + ...\right] + \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w} + regular$$

Vertex operators

$$V_\alpha(z) \equiv : e^{i\alpha\phi(z)} : \quad \Longrightarrow \quad : e^{i\alpha\phi(z)} := \sum_{n=0}^{\infty} \frac{1}{n!} \; : \left(i\alpha\,\phi(z) \right)^n :$$

$$T(z)\,V_\alpha(w) \sim -2\pi g \; : \partial\phi(z)\,\partial\phi(z) : \sum_{n=0}^{\infty}\frac{1}{n!} \; : \left(i\alpha\,\phi(w) \right)^n :=$$

$$-2\pi g\,(i\alpha)^2 \; < \partial\phi(z)\,\phi(w) >^2 \sum_{n=2}^{\infty}\frac{1}{(n-2)!} \; \left(i\alpha\,\phi(w) \right)^{n-2} -$$

$$-2\pi g\,(i\alpha)\,2 \; < \partial\phi(z)\,\phi(w) > \; \partial\phi(z) \sum_{n=1}^{\infty}\frac{1}{(n-1)!} \; \left(i\alpha\,\phi(w) \right)^{n-1}$$

$$T(z)\,V_\alpha(w) \sim \frac{\alpha^2}{8\pi g}\,\frac{V_\alpha(w)}{(z-w)^2} + \frac{\partial V_\alpha(w)}{(z-w)}$$

V_α is a primary field of weight

$$h = \frac{\alpha^2}{8\pi g}$$

and scaling dimension

$$\Delta = \frac{\alpha^2}{4\pi g}$$

This is a purely quantum effect!!

The EM tensor is not a primary

$$T(z)T(w) = (2\pi g)^2 : \partial_z\phi\partial_z\phi : : \partial_w\phi\partial_w\phi :$$

$$\sim (2\pi g)^2 \left[2\frac{1}{(4\pi g)^2}\frac{1}{(z-w)^4} + 4\left(-\frac{1}{4\pi g}\frac{1}{(z-w)^2}\right)\partial_z\phi\partial_w\phi \right]$$

$$\sim \frac{1}{2(z-w)^4} - 4\pi g \frac{1}{(z-w)^2}\partial_w\phi[\partial_w\phi + (z-w)\partial_w^2\phi]$$

$$\sim \frac{1}{2(z-w)^4} + \frac{2}{(z-w)^2}[-2\pi g(\partial_w\phi)^2] + \frac{\partial_w[-2\pi g(\partial_w\phi)^2]}{z-w}$$

$\Downarrow$

$$T(z)T(w) \sim \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

In general

$$T(z)T(w) \sim \frac{1}{2}\frac{c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

$c \rightarrow$ central charge of the CFT

Mode expansions

$\phi(z) \rightarrow$ field of conformal weight h

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n \, z^{-n-h} \qquad \phi_m = \oint_{C_0} \frac{dz}{2\pi i} \, z^{m+h-1} \, \phi(z)$$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} \, L_n \qquad L_n = \oint_{C_0} \frac{dz}{2\pi i} \, z^{n+1} \, T(z)$$

Virasoro algebra

$$[L_n, L_m] = \oint_{C_0} \frac{dw}{2\pi i} \, w^{m+1} [L_n, T(w)] = \oint_{C_0} \frac{dw}{2\pi i} \, w^{m+1} \oint_{C_w} \frac{dz}{2\pi i} \, z^{n+1} T(z) T(w) =$$

$$= \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} \, z^{n+1} \, w^{m+1} \left[\frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right]$$

Use:

$$\oint_{C_w} \frac{dz}{2\pi i} \frac{z^{n+1}}{(z-w)^4} = \frac{1}{6} n(n^2-1) w^{n-2} \qquad \oint_{C_w} \frac{dz}{2\pi i} \frac{z^{n+1}}{(z-w)^2} = (n+1) w^n \qquad \oint_{C_w} \frac{dz}{2\pi i} \frac{z^{n+1}}{z-w} = w^{n+1}$$

$$[L_n, L_m] = \oint_{C_0} \frac{dw}{2\pi i} \left[\frac{c}{12} n(n^2-1) w^{n-2} w^{m+1} + 2(n+1) w^n w^{m+1} T(w) + w^{n+1} w^{m+1} \partial_w T \right] =$$

$$= \frac{c}{12} n(n^2-1) \oint_{C_0} \frac{dw}{2\pi i} w^{n+m-1} + \oint_{C_0} \frac{dw}{2\pi i} [2(n+1) - (n+m+2)] w^{n+m+1} T(w) =$$

$$= \frac{c}{12} n(n^2-1) \delta_{n+m,0} + (n-m) \oint_{C_0} \frac{dw}{2\pi i} w^{n+m+1} T(w)$$

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}$$

Virasoro algebra

Similarly $\phi(z) \longrightarrow$ field of conformal weight h

$$[L_n, \phi(w)] = h(n+1)\, w^n\, \phi(w) + w^{n+1}\, \partial\phi(w)$$

$$[L_n, \phi_m] = \left[n(h-1) - m \right] \phi_{n+m}$$

Representation theory

in vacuum $|0>$ ⟹ $T(z)$ regular on $|0>$ at the infinite past $z=0$

⟹ $T(0)\,|0>=\sum_{n=-\infty}^{+\infty} z^{-n-2}\,L_n\,|0>_{|z=0}$ ⟹ $$L_n\,|0>=0\,, \qquad n\geq -1$$

Primary states

$$|h>=\ \phi(0)\,|0> \qquad\qquad \phi(z)\rightarrow \text{primary of conformal weight } h$$

As $$[\,L_n\,,\,\phi(0)\,]\ =\ 0\ ,\ \ n>0 \qquad\qquad [\,L_0\,,\,\phi(0)\,]\ =\ h\,\phi(0)$$

$$L_n\,|h>=\ 0\ ,\ \ n>0$$
$$L_0\,|h>=\ h\,|h>$$

In string theory
physical states
are primaries with h=1

Closed strings in complex variables

Wick rotate as $\tau \to -i\tau$ and define $z = e^{\tau - i\sigma}\ , \ \bar{z} = e^{\tau + i\sigma}$

$$X^\mu(z,\bar{z}) = X^\mu_R(z) + X^\mu_L(\bar{z})$$

$$X^\mu_R(z) = \frac{x^\mu}{2} - i\frac{\alpha'}{2}p^\mu \ln(z) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}z^{-n}$$

$$\partial X^\mu = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha^\mu_n z^{-n-1}$$

$$\alpha^\mu_m = i\sqrt{\frac{2}{\alpha'}}\oint_{C_0}\frac{dz}{2\pi i}z^m\,\partial X^\mu(z) \qquad \alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu$$

OPEs of coordinates

$$X^\mu(z)\,X^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w)$$

corresponds to $g = \frac{1}{2\pi\alpha'}$

CFT with $c = D$

Scalar primary state created by the vertex operator

$$|k> = :\ e^{ik_\mu X^\mu(0)}\ : |0>$$

$$\alpha_0^\mu \, |k> = \sqrt{\frac{\alpha'}{2}} \, k^\mu \, |k> \quad \Longrightarrow \quad p^\mu \, |k> = k^\mu \, |k>$$

$|k>$ has momentum k^μ

The conformal weight is

$$h = \frac{\alpha'}{4} k^2 \quad \Longrightarrow \quad h = 1 \;\rightarrow\; k^2 = \frac{4}{\alpha'}$$

Mass of the state $M^2 = -k^2$

$$\boxed{M^2 = -\frac{4}{\alpha'}}$$

closed string tachyon

Next level

Consider the tensor state:

$$|\zeta, k> = \lim_{z,\bar{z}\to 0} \zeta_{\mu\nu} \ : \partial X^\mu(z)\ \bar{\partial} X^\nu(\bar{z})\ e^{ik_\rho X^\rho(z,\bar{z})} \ : |0>$$

OPE of T with the vertex operator

$$T(z)\,\zeta_{\mu\nu} \ : \partial X^\mu(w)\ \bar{\partial} X^\nu(\bar{w})\ e^{ik_\rho X^\rho(w,\bar{w})} \sim -\frac{i\alpha'}{2}\ \frac{k^\mu\,\zeta_{\mu\nu}}{(z-w)^3}\ : \bar{\partial} X^\nu(\bar{w})\, e^{ik_\rho X^\rho(w,\bar{w})} : +$$

$$+\left[\frac{\frac{\alpha'}{4}k^2 + 1}{(z-w)^2} + \frac{\partial_w}{z-w}\right] \zeta_{\mu\nu} \ : \partial X^\mu(w)\ \bar{\partial} X^\nu(\bar{w})\ e^{ik_\rho X^\rho(w,\bar{w})}$$

The vertex is primary with h=1 if:

$$M^2 = -k^2 = 0$$ ⟶ massless particles

$$k^\mu\,\zeta_{\mu\nu} = 0$$ ⟶ polarization condition