Lecture 6: The superstring

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STRING THEORY

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Motivation

There are several reasons to tackle superstring theory:

- Spectrum of the bosonic string has a tachyon; unstable (*false*?) vacuum.
- Nature has fermions.

We have to modify the (field content of the) world-sheet theory.

If we want to reproduce the Dirac equation for spacetime fermions,

$$i\,\Gamma^{\mu} p_{\mu} + m = 0$$

we must include anticommuting world-sheet fields, ψ^{μ}, with the expectation that their center of mass modes are Γ^{μ}, such that $\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2\eta^{\mu\nu}$.

Notice that if this is the case, a kinetic term of the form

$$\psi^{\mu}\bar{\partial}\psi_{\mu} \qquad \Rightarrow \qquad \text{Cliff}_D \text{ algebra}$$

Thus, D-dim Poincaré invariance makes the 2d world-sheet theory SUSY.

The superstring action

The equation of motion for spacetime spinors arises from the conserved supercurrent of the 2d theory.

Start with the gauge-fixed Polyakov action; add world-sheet Majorana spinors,

$$S = -\frac{T}{2}\int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar\sigma} d\sigma \left(\eta^{\alpha\beta}\partial_\alpha x^\mu\,\partial_\beta x_\mu - i\,\overline{\psi}^\mu \rho^\alpha\partial_\alpha\psi_\mu\right)$$

Here ρ^α are 2d Dirac matrices, $\{\rho^\alpha,\rho^\beta\} = -2\eta^{\alpha\beta}$; a convenient basis being,

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

We refer to the *real* ($\overline{\chi} = \chi^t\,\rho^0$, $\overline{\chi}\psi = \overline{\psi}\chi$) components of ψ in this basis as:

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \qquad \psi_\pm^\star = \psi_\pm$$

It is important to emphasize that at this stage it is not obvious at all that such theory will lead to spacetime fermions or spacetime supersymmetry; in fact, the 2d fermion fields are bosons with respect to the spacetime Lorentz group.

Holomorphicity of Majorana-Weyl spinors

The fermionic kinetic term becomes

$$\overline{\psi} \cdot \rho^\alpha \partial_\alpha \psi = \psi_- \cdot \partial_+ \psi_- + \psi_+ \cdot \partial_- \psi_+$$

where we recall that the light-cone derivatives $\partial_\pm$ are

$$\partial_\pm = \frac{\partial}{\partial \xi^\pm} = \frac{1}{2} (\partial_\tau \pm \partial_\sigma) \qquad \xi^\pm = \tau \pm \sigma$$

The equation of motion for the fermions is massless 2d Dirac equation,

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0$$

(still needs boundary conditions). Thus, the Majorana-Weyl fermions:

- $\psi_-^\mu = \psi_-^\mu(\xi^-)$ describe right-movers while
- $\psi_+^\mu = \psi_+^\mu(\xi^+)$ describe left-movers.

The equations of motion and constraints for x^μ are the same as before.

World-sheet supersymmetry

The system of free scalars and fermions is automatically invariant under the global, infinitesimal, world-sheet supersymmetry transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon}\,\psi^\mu \qquad \delta_\epsilon \psi^\mu = -i\,\rho^\alpha \partial_\alpha X^\mu\,\epsilon$$

with ϵ a constant, anticommuting two-component spinor. Indeed,

$$\delta S = -T\int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma \left(\eta^{\alpha\beta}\partial_\alpha \delta_\epsilon X \cdot \partial_\beta X - \frac{i}{2}\left[\delta_\epsilon \overline{\psi}\cdot \rho^\alpha \partial_\alpha \psi + \overline{\psi}\cdot \rho^\alpha \partial_\alpha \delta_\epsilon \psi\right]\right)$$

$$= -T\int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma \left(\eta^{\alpha\beta}\,\bar{\epsilon}\,\partial_\alpha \psi + \frac{1}{2}\left[\bar{\epsilon}\,\rho^\beta \rho^\alpha \partial_\alpha \psi - \overline{\psi}\,\rho^\alpha \rho^\beta \epsilon\,\partial_\alpha\right]\right)\cdot \partial_\beta X$$

$$= -T\int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma \left[\left(\eta^{\alpha\beta}\,\bar{\epsilon}\,\partial_\alpha \psi + \frac{1}{2}\bar{\epsilon}\left\{\rho^\beta, \rho^\alpha\right\}\partial_\alpha \psi\right)\cdot \partial_\beta X - \bar{\epsilon}\,\partial_\alpha \mathcal{J}^\alpha\right]$$

where $\partial_\alpha \overline{\psi}\,\rho^\alpha \rho^\beta \epsilon = \bar{\epsilon}\,\rho^\alpha \rho^\beta \partial_\alpha \psi$; δS is given by the Noether supercurrent

$$\delta S = T\int_{\tau_1}^{\tau_2} d\tau \int_0^{\bar{\sigma}} d\sigma\,\bar{\epsilon}\,\partial_\alpha \mathcal{J}^\alpha = 0 \qquad \mathcal{J}^\alpha = \frac{1}{2}\rho^\beta \rho^\alpha \psi \cdot \partial_\beta X$$

Some components of the supercurrent vanish. Indeed, due to $\rho^\alpha \rho^\beta \rho_\alpha = 0$,

$$\rho^\alpha \mathcal{J}_\alpha = 0 \qquad \text{only one chirality survives} : J_\alpha$$

which in light-cone coordinates read

$$J_\pm = \psi_\pm \cdot \partial_\pm x$$

and they are conserved

$$\partial_- J_+ = \partial_+ J_- = 0$$

We can use the Noether method to compute the energy-momentum tensor,

$$T_{\pm\pm} = \partial_\pm x \cdot \partial_\pm x + \frac{i}{2} \psi_\pm \cdot \partial_\pm \psi_\pm \qquad T_{+-} = T_{-+} = 0$$

(the trace still vanishes). The Virasoro constraints $T_{++} = T_{--} = 0$ where used in the bosonic case to eliminate ghosts (negative norm states) coming from x^0. What about those coming from ψ^0?

Boundary conditions and mode expansion: open strings

To remove the time-like component ψ^0, we must also impose $J_+ = J_- = 0$. These come from the locally supersymmetric ungauged Polyakov action.

The mode decomposition and boundary conditions for $x^\mu(\tau, \sigma)$ are as before.

Let us consider the free fermionic fields $\psi^\mu(\tau, \sigma)$. In the case of open strings, the variational principle for the Polyakov action requires

$$\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_- = 0 \qquad \text{at} \quad \sigma = 0, \pi$$

The Dirac equation admits two possible boundary conditions consistent with Lorentz invariance, namely

$$\psi_+^\mu = \pm\, \psi_-^\mu \qquad (\text{and hence } \delta\psi_+^\mu = \pm\, \delta\psi_-^\mu) \quad \text{at each end}$$

The overall relative sign between ψ_-^μ and ψ_+^μ is irrelevant; we take, without loss of generality,

$$\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$$

This still leaves two possibilities at the other endpoint of the string, $\sigma = \pi$.

Boundary conditions and mode expansion: open strings

According to the relative sign,

$$\psi_+^\mu(\tau,\pi) = +\psi_-^\mu(\tau,\pi) \qquad \text{Ramond (R) boundary condition}$$

$$\psi_+^\mu(\tau,\pi) = -\psi_-^\mu(\tau,\pi) \qquad \text{Neveu – Schwarz (NS) boundary condition}$$

The mode expansion of the Dirac equation becomes

$$\psi_\pm^\mu(\tau\pm\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}} \psi_r^\mu \ \epsilon^{-i\,r(\tau\pm\sigma)} \qquad \text{R – boundary condition}$$

$$\psi_\pm^\mu(\tau\pm\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} \psi_r^\mu \ \epsilon^{-i\,r(\tau\pm\sigma)} \qquad \text{NS – boundary condition}$$

The Majorana condition requires

$$\psi_{-r}^\mu = (\psi_r^\mu)^\star$$

Note: Only the R sector gives a zero mode ψ_0^μ (that should lead to fermions).

Boundary conditions and mode expansion: closed strings

For closed strings the surface terms vanish when the boundary conditions are periodic or anti-periodic, separately, for each component of $\psi^\mu_\pm$,

$$\psi^\mu_-(\tau-\sigma) = \sum_r \psi^\mu_r \, \epsilon^{-2\,i\,r(\tau-\sigma)} \qquad \psi^\mu_+(\tau+\sigma) = \sum_r \bar\psi^\mu_r \, \epsilon^{-2\,i\,r(\tau+\sigma)}$$

with the mode index r constrained according to either R or NS conditions. The bar on $\bar\psi^\mu_r$ shouldn't be confused with the one in Dirac's equation.

Corresponding to the different pairings between left- and right-moving modes, there are four distinct closed string sectors that can be grouped according to the space-time nature of their quantum states:

$$\mathrm{R-R} \quad \text{and} \quad \mathrm{NS-NS} \qquad (\text{bosons})$$

$$\mathrm{R-NS} \quad \text{and} \quad \mathrm{NS-R} \qquad (\text{fermions})$$

The superstring has, as we will shortly prove, space-time fermions!

Not surprisingly, they come from the *odd* combination of boundary conditions.

Super Virasoro constraints: closed strings

The closed string mode expansion of the physical constraints are given by

$$T_{--}(\xi^-) = \alpha' \sum_{n=-\infty}^{\infty} L_n \, \epsilon^{-2i\,n\,\xi^-} \qquad T_{++}(\xi^+) = \alpha' \sum_{n=-\infty}^{\infty} \bar{L}_n \, \epsilon^{-2i\,n\,\xi^+}$$

as well as for the current constraints:

$$J_-(\xi^-) = \alpha' \sum_r G_r \, \epsilon^{-2\,i\,r\,\xi^-} \qquad J_+(\xi^+) = \alpha' \sum_r \bar{G}_r \, \epsilon^{-2\,i\,r\,\xi^+}$$

where, inverting the above expressions, the super-Virasoro generators read

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m + \frac{1}{4} \sum_r (2r - n) \, \psi_{n-r} \cdot \psi_r$$

$$G_r = \sum_{m=-\infty}^{\infty} \alpha_m \cdot \psi_{r-m} \qquad \text{(analogous for } \bar{L}_n \text{ and } \bar{G}_r\text{)}$$

These expressions result from mode expansion of the world-sheet fields.

Normal ordering and zero-point energy

Going quantum, we have to be careful about normal ordering of operators:

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} :\alpha_{n-m} \cdot \alpha_m: + \frac{1}{4} \sum_r (2r - n) \; :\psi_{n-r} \cdot \psi_r:$$

an ambiguity arising for L_0 (and $\bar{L}_0$).

As before, the constraints must be imposed through a *physical state condition*

$$L_{m>0} \, |\mathrm{phys}\rangle = 0 \qquad G_{r>0} \, |\mathrm{phys}\rangle = 0 \qquad (L_0 - a) \, |\mathrm{phys}\rangle = 0$$

(same for $\bar{L}_m$ and $\bar{G}_r$) for a given *c*-number a that we shall determine.

This computation, as for the bosonic string, is better performed in space-time light-cone coordinates

$$x^{\pm} := \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}) \qquad \psi^{\pm} := \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^{D-1})$$

while x^i and ψ^i are the remaining space-like coordinates of the superstring.

Light-cone gauge

Recall that in the bosonic string, as we will do now, we gauged away α_n^+ by choosing the light-cone gauge $\tau \sim x^+$. We shall take at the same time $\psi^+ = 0$ if we want to be consistent with world-sheet supersymmetry.

Given that $\partial_+ x^+ = \frac{1}{2} p^+$ and $\psi^+ = 0$, we can solve for x^- and ψ^-

$$\partial_+ x^- = \frac{1}{\alpha' p^+} \left(\partial_+ x^i \, \partial_+ x^i + \frac{i}{2} \psi^i \, \partial_+ \psi^i \right) \qquad \psi^- = \frac{2}{\alpha' p^+} \psi^i \partial_+ x^i$$

in terms of the space-like components. This implies for the oscillators

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'} p^+} \left[\sum_{m=-\infty}^{\infty} : \alpha^i_{n-m} \, \alpha^i_m : + \frac{1}{2} \sum_r (2r - n) \, : \psi^i_{n-r} \, \psi^i_r : -2a \, \delta_n \right]$$

$$\psi_r^- = \sum_{m=-\infty}^{\infty} \alpha^i_m \, \psi^i_{r-m} \qquad \text{(analogous for } \bar{\alpha}_n^- \text{ and } \bar{\psi}_r^- \text{)}$$

The same computations can be performed in the case of open strings.

Super Virasoro algebra

By relying in the commutation relation of the x^μ and the anti-commutators

$$\{\psi_r^\mu \,,\, \psi_s^\nu\} = \delta_{r+s,0}\, \eta^{\mu\nu}$$

it can be shown that L_n and G_r generate the $\mathcal{N} = 1$ supersymmetric extension of the Virasoro algebra:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} A(n)\, \delta_{n+m,0}$$

$$[L_n, G_r] = \frac{1}{2}\Big(n - 2r\Big)\, G_{n+r} \qquad \Rightarrow \qquad \mathrm{Osp}(1|2) \text{ subalgebra for NS}$$

$$\{G_r, G_s\} = 2\, L_{r+s} + \frac{c}{3} B(r)\, \delta_{r+s,0}$$

- $A(n) = n^3 - n$ and $B(r) = r^2 - 1/4$ for the NS sector,
- $A(n) = n^3$ and $B(r) = r^2$ for the R sector.

$c = D + \frac{D}{2}$ is the total contribution to the conformal anomaly.

Super Virasoro algebra: the conserved angular momentum

The densities of momentum and angular momentum along the string are Noether currents

$$P^\mu_\alpha = \partial_\alpha x^\mu \qquad J^{\mu\nu}_\alpha = x^\mu \partial_\alpha x^\nu - x^\nu \partial_\alpha x^\mu + i\bar{\psi}^\mu \rho_\alpha \psi^\nu$$

Inserting the mode expansions for x^μ and ψ^μ

$$J^{\mu\nu} = \int_0^\pi J^{\mu\nu}_\tau d\sigma = J^{\mu\nu}_B + J^{\mu\nu}_F = \ell^{\mu\nu} + E^{\mu\nu} + J^{\mu\nu}_F$$

where

$$\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \qquad E^{\mu\nu} = -i \sum_{n=1}^\infty \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right)$$

$$J^{\mu\nu}_F = -i \sum_{r \geq 0} \left(\psi^\mu_{-r} \psi^\nu_r - \psi^\nu_{-r} \psi^\mu_r \right) \qquad \text{NS}$$

$$J^{\mu\nu}_F = -\frac{i}{2}[\psi^\mu_0, \psi^\nu_0] - i \sum_{k=1}^\infty \left(\psi^\mu_{-k} \psi^\nu_k - \psi^\nu_{-k} \psi^\mu_k \right) \qquad \text{R}$$

Mass spectrum of the open string

As in the bosonic string case, we can define a number operator as

$$N := \sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m + \sum_{r>0} r\,\psi^i_{-r}\,\psi^i_r$$

$$L_0 = -\alpha'\,M^2 + N \qquad \Rightarrow \qquad M^2 = \frac{1}{\alpha'}(n-a)$$

since $(L_0 - a)\,|\text{phys}\rangle = 0$ and $N\,|\text{phys}\rangle = n\,|\text{phys}\rangle$.

The open string spectrum of states has two independent NS and R sectors. Let us start by analyzing the NS sector.

The NS Fock-space unique nondegenerate ground state $|k;0\rangle_{\text{NS}}$ satisfies

$$\alpha^\mu_{m>0}|k;0\rangle_{\text{NS}} = \psi^\mu_{r>0}|k;0\rangle_{\text{NS}} = 0 \qquad \alpha^\mu_0|k;0\rangle_{\text{NS}} = \sqrt{2\alpha'}\,k^\mu|k;0\rangle_{\text{NS}}$$

Since $N\,|k;0\rangle_{\text{NS}} = 0$, this vacuum state would be tachyonic for $a > 0$:

$$M^2 = -\frac{a}{\alpha'} \qquad \text{as for the bosonic string!}$$

Determination of the normal ordering constant

In order to evaluate a, we must reorder L_0:

$$L_0 = \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha^i_{-m}\,\alpha^i_m + \frac{1}{2}\sum_{r=-\infty}^{\infty}\left(r+\frac{1}{2}\right)\psi^i_{-r-1/2}\,\psi^i_{r+1/2}$$

Taking into account that $[\alpha^i_m,\,\alpha^j_n] = m\,\delta^{ij}\,\delta_{mn}$ and $\{\psi^i_{-s},\,\psi^j_t\} = \delta^{ij}\,\delta_{st}$,

$$L_0 = -\alpha'\,M^2 + N + \frac{D-2}{2}\sum_{m=1}^{\infty} m - \frac{D-2}{2}\sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right)$$

Thus, we conclude that, since $(L_0 - a)\,|\text{phys}\rangle = 0$,

$$a_{\rm NS} = -\frac{D-2}{2}\left[\sum_{m=1}^{\infty} m - \sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right)\right]$$

in the NS sector. This expression demands regularization.

However, we can realize at this point that $a_{\rm R} = 0$ (the magic of SUSY!).

Regularized infinite sum

It is useful to compute in general (for $\alpha > 0$; for the bosonic string, $\alpha = 0$),

$$Z_\alpha = \frac{1}{2}\sum_{n=0}^{\infty}(n+\alpha) = -\frac{1}{24} + \frac{1}{4}\alpha(1-\alpha)$$

as seen from the $\epsilon \to 0$ limit of the finite part of

$$Z_\alpha(\epsilon) = \frac{1}{2}\sum_{n=0}^{\infty}(n+\alpha)\,e^{-(n+\alpha)\epsilon} = -\frac{1}{2}\frac{d}{d\epsilon}\sum_{n=0}^{\infty}e^{-(n+\alpha)\epsilon} = -\frac{1}{2}\frac{d}{d\epsilon}\left(\frac{e^{-\alpha\epsilon}}{1-e^{-\epsilon}}\right)$$

$$= -\frac{1}{2}\frac{d}{d\epsilon}\left[\left(1-\alpha\epsilon+\frac{1}{2}\alpha^2\epsilon^2+\mathcal{O}(\epsilon^3)\right)\left(\frac{1}{\epsilon}+\frac{1}{2}+\frac{1}{12}\epsilon+\mathcal{O}(\epsilon^2)\right)\right]$$

$$= -\frac{1}{2}\frac{d}{d\epsilon}\left[\frac{1}{\epsilon}+\frac{1}{2}(1-2\,\alpha)+\frac{1}{12}(1-6\,\alpha+6\,\alpha^2)\,\epsilon+\mathcal{O}(\epsilon^2)\right]$$

$$= \frac{1}{2\,\epsilon^2} - \frac{1}{24} + \frac{1}{4}\alpha(1-\alpha) + \mathcal{O}(\epsilon)$$

The critical dimension

With this regularization we can confirm that $a_{\rm R} = 0$, and

$$a_{\rm NS} = (D-2)\left(Z_{1/2} - Z_0\right) = \frac{D-2}{16} \qquad \Rightarrow \qquad a_{\rm NS} = \frac{1}{2}$$

The tachyonic vacuum of the NS sector   see also GSW, end of § 4.2.2 and § 4.3.1

$$M^2 = -\frac{D-2}{16\,\alpha'}$$

The first excited level can be constructed by acting with $\psi^i_{-\frac{1}{2}}$ on $|k;0\rangle_{\rm NS}$,

$$N\,\left(\psi^i_{-\frac{1}{2}}\,|k;0\rangle_{\rm NS}\right) = \frac{1}{2}\,\left(\psi^i_{-\frac{1}{2}}\,|k;0\rangle_{\rm NS}\right)$$

Thus, its mass can be readily determined to be

$$M^2 = \frac{1}{2\,\alpha'}\left(1 - \frac{D-2}{8}\right) \qquad \Rightarrow \qquad D = 10$$

This vector has $D-2$ components; in a relativistic theory it must be massless (describe the physical polarizations of an open string photon field $A_\mu(x)$).

The open string spectrum

The NS sector of the RNS superstring is quite reminiscent of the bosonic string spectrum.

Indeed, all NS states are space-time bosons: they transform in appropriate irreducible representations of $SO(8)$, which is the Little group of $SO(1,9)$.

In the R sector there are zero modes ψ_0^μ which satisfy the 10d Dirac algebra!

$$\{\psi_0^\mu\,, \psi_0^\nu\} = \eta^{\mu\nu}$$

As announced, the zero modes can be regarded as Dirac matrices,

$$\psi_0^\mu = \frac{1}{\sqrt{2}}\,\Gamma^\mu \qquad \Rightarrow \qquad \text{degenerate ground state}$$

This is the origin of space-time fermions in the superstring.

In particular they are finite dimensional operators: all states in the R sector should be space-time fermions in order to furnish representation spaces on which these operators can act.