Lecture 7: GSO projection and space-time supersymmetry

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STRING THEORY

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Summary from Lecture 6

The open superstring includes world-sheet fermionic fields $\psi^\mu(\tau,\sigma)$ admitting

$$\begin{aligned}\psi_+^\mu(\tau,\pi) &= +\psi_-^\mu(\tau,\pi) \qquad \text{Ramond (R) boundary condition}\\ \psi_+^\mu(\tau,\pi) &= -\psi_-^\mu(\tau,\pi) \qquad \text{Neveu – Schwarz (NS) boundary condition}\end{aligned}$$

Correspondingly, the mode expansion is different for both sectors: only the R sector has a zero mode ψ_0^μ (that leads to fermions).

For closed strings, the different pairings between left- and right-moving modes lead to four distinct closed string sectors:

$$\mathrm{R-R}\ \&\ \mathrm{NS-NS}\ \ (\text{bosons}) \qquad \text{and} \qquad \mathrm{R-NS}\ \&\ \mathrm{NS-R}\ \ (\text{fermions})$$

The NS ground state $|k;0\rangle_{\rm NS}$ is tachyonic,

$$a_{\rm NS} = \frac{D-2}{16} \qquad \Rightarrow \qquad M^2 = -\frac{D-2}{16\,\alpha'}$$

Summary from Lecture 6

The first excited level, $\psi^i_{-\frac{1}{2}}\,|k;0\rangle_{\mathrm{NS}}$, is $A_\mu(x)$ thus $D_{\mathrm{critical}} = 10$.

The NS sector of the RNS superstring is quite reminiscent of the bosonic string spectrum, with $a_{\mathrm{NS}} = 1/2$.

Indeed, all NS states are space-time bosons: they transform in appropriate irreducible representations of $SO(8)$, which is the Little group of $SO(1,9)$.

In the R sector, $a_{\mathrm{R}} = 0$, and there are zero modes ψ_0^μ which satisfy the 10d Dirac algebra!

$$\left\{\psi_0^\mu\,,\,\psi_0^\nu\right\} = \eta^{\mu\nu} \qquad \Rightarrow \qquad \psi_0^\mu = \frac{1}{\sqrt{2}}\,\Gamma^\mu$$

This is the origin of space-time fermions in the superstring.

In particular they are finite dimensional operators: all states in the R sector should be space-time fermions in order to furnish representation spaces on which these operators can act.

More on space-time fermions

The zero mode part of the fermionic constraint gives a wave equation for a fermionic string in the R sector known as the Dirac-Ramond equation

$$G_0 \, |\text{phys}\rangle_{\text{R}} = \left(\alpha_0 \cdot \psi_0 + \sum_{n \neq 0} \alpha_{-n} \cdot \psi_n \right) |\text{phys}\rangle_{\text{R}} = 0$$

Notice that the zero mode piece $\alpha_0 \cdot \psi_0$ is precisely the space-time Dirac operator, $\partial\!\!\!/ = \Gamma^\mu \, \partial_\mu$, in momentum space (since $\alpha_0^\mu \sim p^\mu$ and $\psi_0^\mu \sim \Gamma^\mu$).

Thus the R sector fermionic ground state $|\psi^{(0)}\rangle_{\text{R}}$, defined by

$$\alpha_n^\mu \, |\psi^{(0)}\rangle_{\text{R}} = \psi_n^\mu \, |\psi^{(0)}\rangle_{\text{R}} = 0 \qquad n > 0$$

satisfies the massless Dirac wave equation in ten dimensions.

It has to be a degenerate ground state: the action of ψ_0^i costs no energy. We cannot impose $\psi_0^\mu \, |\psi^{(0)}\rangle_{\text{R}} = 0$ due to the Clifford algebraic structure.

Now, a spinor in 10d is expected to have $2^{[D/2]} = 32$ complex components.

A Majorana representation, *i.e.*, a Clifford algebra where all Γ^μ are imaginary, is possible in 10d. Thus, the Majorana spinor has 32 real components.

The Weyl condition can be simultaneously imposed on the spinor, through the chirality matrix $\Gamma_{11} := \Gamma^0\,\Gamma^1 \cdots \Gamma^9$,

$$\{\Gamma_{11}, \Gamma^\mu\} = 0 \qquad \text{and} \qquad (\Gamma_{11})^2 = 1$$

The latter requires a subtle cancelation of 54 minus signs.

Spinors Ψ obeying $\Gamma_{11}\,\Psi = \pm\Psi$ are chiral fermions. Notice that Γ_{11} is real (in contrast to what happens in 4d). Thus, the spinor can be Majorana and Weyl, at the same time, having 16 real components.

Now, the massless Dirac equation, $\Gamma \cdot \partial\Psi = 0$ implies that eight components can be related on-shell to the other eight. These are the 8 real components of $|\psi^{(0)}\rangle_{\mathrm{R}}$ (that we find in the light-cone gauge).

There are two space-time chiralities and, as well, we shall see two R ground states in string theory:

$$|\psi^{(0)}_{(\pm)}\rangle_{\mathrm{R}} \qquad \text{such that} \qquad \Gamma_{11}\,|\psi^{(0)}_{(\pm)}\rangle_{\mathrm{R}} = \pm|\psi^{(0)}_{(\pm)}\rangle_{\mathrm{R}}$$

Light states and level matching conditions

How do these two R chiral ground states can be constructed? Indeed, we can build such representations by considering

$$\ell_k^{\pm} := \psi_0^{2k} \pm i\,\psi_0^{2k-1} \qquad k = 1, \ldots, 4$$

Then we define a lowest weight state by $\ell_k^- |0\rangle_{\rm R} = 0$ and build the set of states by application of the ℓ_k^+ operators:

$$\{|0\rangle_{\rm R},\ \ell_{k_1}^+\,\ell_{k_2}^+\,|0\rangle_{\rm R},\ \ell_1^+\,\ell_2^+\,\ell_3^+\,\ell_4^+\,|0\rangle_{\rm R}\} \qquad \{\ell_k^+\,|0\rangle_{\rm R},\ \ell_{k_1}^+\,\ell_{k_2}^+\,\ell_{k_3}^+\,|0\rangle_{\rm R}\}$$

Both multiplets contain eight states. They correspond to both possible chiral ground states, $|\psi_{(\pm)}^{(0)}\rangle_{\rm R}$, transforming in the two inequivalent, irreducible, MW representations of $SO(8)$, $\mathbf{8_S} = \mathbf{8}_+$ and $\mathbf{8}_-$.

The Hilbert space in the R sector is obtained by acting with the negative modding operators on these ground states in all possible ways.

All in all, the number of degrees of freedom of the open string massless multiplet corresponds to an Abelian Yang-Mills supermultiplet.

The closed superstring spectrum: massless states

We can choose the NS or R sector independently for left and right movers in the closed superstring. The spectrum is obtained by taking tensor products of left- and right-movers, and by using appropriate *gluing constraints* due to both modular and Poincaré invariances. Then, there are four distinct sectors:

SECTOR	STATISTICS	$SO(8)$ REPRESENTATION	MASSLESS
NS-NS	boson	$\mathbf{8_V} \otimes \mathbf{8_V} = \mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1}$	$g_{\mu\nu}$, $B_{\mu\nu}$, Φ
NS-R	fermion	$\mathbf{8_V} \otimes \mathbf{8_S} = \mathbf{8_S} \oplus \mathbf{56_S}$	Ψ_μ , λ
R-NS	fermion	$\mathbf{8_S} \otimes \mathbf{8_V} = \mathbf{8_S} \oplus \mathbf{56_S}$	Ψ'_μ , λ'
R-R	boson	$\mathbf{8_S} \otimes \mathbf{8_S} = \mathrm{p-forms}$	R-R fields

All *massless* states, which are picked up in the limit $\alpha' \to 0$, are summarized according to their properties under $SO(8)$. The closed superstring sector leaves us with space-time supergravity fields.

The GSO projection

The superstring spectrum admits a consistent truncation, the GSO projection, $P_{\rm GSO}$, which is necessary for well-definiteness of the interacting theory.

• NS sector: $P_{\rm GSO}$ acts by keeping states with an odd number of ψ oscillator excitations. This amounts to imposing on physical states,

$$|{\rm phys}\rangle_{\rm NS} \qquad \rightarrow \qquad P_{\rm GSO}\,|{\rm phys}\rangle_{\rm NS}$$

the projector operator,

$$P_{\rm GSO} := \frac{1}{2}\left(1-(-1)^F\right) \qquad \text{with} \qquad F := \sum_{r>0} \psi_{-r}\cdot\psi_r$$

F being the *fermion number operator* which obeys

$$\left\{(-1)^F\,,\,\psi^\mu\right\} = 0$$

Thus only half-integer values of the level number are possible. This washes out the tachyon and leads to a bosonic ground state which is massless.

The GSO projection

- R sector: The ground state is an eigenstate of Γ_{11}. The generalization to massive states must be non-trivial since the Weyl condition should not work.

The generalization is given by the Klein operator, $\bar{\Gamma}$,

$$\bar{\Gamma} := \Gamma^{11}\,(-1)^{\sum_{r=1}^{\infty}\psi_{-r}\cdot\psi_r}$$

which has the property

$$\{\,\bar{\Gamma}, \psi_r^\mu\} = 0$$

since the factor Γ_{11} anticommutes with $\psi_0^\mu \simeq \Gamma^\mu$ and the remaining factor in $\bar{\Gamma}$ anticommutes with the remaining modes ψ_r^μ, $r \neq 0$.

Indeed, this operator plays the role of $(-1)^F$ in the R sector,

$$\{(-1)^F, \psi^\mu(\sigma,\tau)\} = 0$$

The GSO projection in the R sector is thus

$$\bar{\Gamma}\,|\psi_{(\pm)}\rangle_{\mathrm{R}} = \pm|\psi_{(\pm)}\rangle_{\mathrm{R}}$$

Either sign leads to the Abelian Yang-Mills supermultiplet as a ground state.

The GSO projection: the massive states

Any state in the R sector can be built as

$$\psi^{i_1}_{-r_1} \cdots \psi^{i_n}_{-r_n} |\psi^{(0)}_{(\pm)}\rangle_{\rm R}$$

a suitable product, possibly including zero-mode operators. It can be seen that

$$(-1)^F \, \psi^{i_1}_{-r_1} \cdots \psi^{i_n}_{-r_n} |\psi^{(0)}_{(\pm)}\rangle_{\rm R} = (-1)^n \, \psi^{i_1}_{-r_1} \cdots \psi^{i_n}_{-r_n} |\psi^{(0)}_{(\pm)}\rangle_{\rm R}$$

This is not the Weyl condition: Γ^{11} does not anticommute with $i\,\Gamma \cdot \partial + m$.

Strikingly, in string theory

$$\{\bar{\Gamma}, G_0\} = 0$$

is an exact statement: the GSO projection can be applied to massive levels.

Consider the first excited fermionic level. The possible states are

$$\alpha^i_{-1} \, |\psi^{(0)}_{(+)}\rangle_{\rm R} \qquad \text{and} \qquad \psi^i_{-1} \, |\psi^{(0)}_{(-)}\rangle_{\rm R}$$

These combine to give a Majorana spinor: irreducible massive representation of the Lorentz group.

The GSO projection and space-time supersymmetry

The spectrum of allowed physical masses are integral multiples of α'^{-1}. In particular, the bosonic ground state is now massless, and the spectrum no longer contains a tachyon (which has fermion number $F = 0$).

The GSO projection also removes from the NS sector the weird space-time bosons built with an odd number of anticommuting world-sheet fields acting, for instance, on a massless vector meson.

A third virtue of the GSO projection: it leads to space-time supersymmetry.

Besides the formal interest, the massless spin 3/2 particle should couple to a current whose conserved charge is spin 1/2, *i.e.*, a supersymmetric charge.

Let us give some circumstancial evidence that the RNS string with the GSO projection is a supersymmetric theory in ten dimensions.

We have already seen that the ground state of the open string spectrum is a massless vector superfield.

How can we check whether this is true or not for the full spectrum?

Density of states for the bosonic string

Let us go back to the bosonic string. We wish to know the density of states for the massive levels, labeled by the eigenvalue of the number operator,

$$N\,|\psi_n\rangle := \sum_{i=1}^{D-2}\sum_{k=1}^{\infty} \alpha^i_{-k}\,\alpha^i_k\,|\psi_n\rangle = n\,|\psi_n\rangle$$

The total number of open string states with $\alpha'\,M^2 = n-1$, denoted by d_n, is conveniently defined as the coefficient of w^n in the generating function

$$G(w) := \mathrm{tr}_{\mathcal{H}}\,w^N = \sum_{n=0}^{\infty} d_n\,w^n \quad \Rightarrow \quad d_n = \frac{1}{2\pi i}\oint \frac{G(w)}{w^{n+1}}\,dw$$

$w \neq 1$ being a complex number (indeed, $|w| \neq 1$). Now, these are $D-2$ copies of the harmonic oscillator $\{\alpha_{-k}, \alpha_k\}$,

$$G(w) = \mathrm{tr}_{\mathcal{H}}\,w^N = \prod_{k=1}^{\infty}\left(\mathrm{tr}\,w^{\,\alpha_{-k}\,\alpha_k}\right)^{D-2} = \prod_{k=1}^{\infty}\left(\sum_{n=0}^{\infty} w^{\,k\,n}\right)^{D-2}$$

Contributions come from $|n,k\rangle \equiv (\alpha_{-k})^n\,|0\rangle$, since $\alpha_{-k}\,\alpha_k\,|n,k\rangle = n\,k\,|n,k\rangle$.

Therefore:

$$G(w) = \prod_{k=1}^{\infty} \left(\sum_{n=0}^{\infty} w^{k\,n} \right)^{D-2} = \prod_{k=1}^{\infty} \left(\frac{1}{1 - w^k} \right)^{D-2} = [f(w)]^{-24}$$

where we already introduced the bosonic critical dimension and

$$f(w) := \prod_{n=1}^{\infty} (1 - w^n)$$

which is called the classical partition function.

For any complex number with positive imaginary part, τ,

$$\eta(\tau) \equiv e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

is the Dedekind eta function. It satisfies the following modular properties:

$$\eta(-1/\tau) = (-i\tau)^{1/2}\,\eta(\tau) \qquad \text{and} \qquad \eta(\tau + 1) = e^{\pi i/12}\,\eta(\tau)$$

Density of states for the bosonic string

Applied to $f(w)$, this gives the Hardy-Ramanujan formula

$$f(w) = \left(\frac{-2\pi}{\log w}\right)^{1/2} w^{-1/24}\, q^{1/12}\, f(q^2) \qquad \text{where} \quad q = \exp\left(\frac{-2\pi^2}{\log w}\right)$$

For $w \to 1$,

$$f(w) \sim A(1-w)^{-1/2} \exp\left(-\frac{\pi^2}{6(1-w)}\right)$$

Having found the generating function $G(w)$ it is now easy to grasp

$$d_n = \frac{1}{2\pi i} \oint \frac{G(w)}{w^{n+1}}\, dw$$

where the contour of integration is a small circle about the origin.

It can be seen that, for large n, the density of states grows dramatically:

$$d_n \propto n^{-27/4}\, e^{4\pi\sqrt{n}}$$

Density of states for the RNS superstring

We want to make a similar computation for the RNS superstring. The only difference is that we have to take into account the action of $(-1)^F$ in both sectors of the open string.

The total number of open string states with $\alpha' M^2 = n$, denoted by $d_n^{\rm NS}$, is conveniently defined through the generating function

$$G_{\rm NS}(w) := \frac{1}{\sqrt{w}}\, {\rm tr}_{\mathcal{H}}\, P_{\rm GSO}\, w^N = \frac{1}{2\sqrt{w}}\, {\rm tr}_{\mathcal{H}} \left(1 - (-1)^F\right) w^N = \sum_{n=0}^{\infty} d_n^{\rm NS}\, w^n$$

where we considered that the massless level has number eigenvalue $1/2$.

The new feature is the factor of the trace over ψ_r^i: since each fermionic state is either occupied or unoccupied, it gives $(1 + w^r)$ for each mode. However, the presence of $P_{\rm GSO}$ changes the sign of the occupied states.

The result for the complete trace is

$$G_{\rm NS}(w) = \frac{1}{2\sqrt{w}} \left[\prod_{n=1}^{\infty} \left(\frac{1 + w^{n-1/2}}{1 - w^n} \right)^8 - \prod_{n=1}^{\infty} \left(\frac{1 - w^{n-1/2}}{1 - w^n} \right)^8 \right]$$

Density of states for the RNS superstring

The degeneracy of the fermionic levels is given in a similar manner by

$$G_{\rm R}(w) := {\rm tr}_{\mathcal{H}}\, P_{\rm GSO}\, w^N = \sum_{n=0}^{\infty} d_n^{\rm R}\, w^n = 8\, {\rm tr}_{\mathcal{H}}\, w^N$$

where, of course, now $P_{\rm GSO} = \frac{1}{2}(1 + \bar{\Gamma})$.

There is a factor of 8 representing the degeneracy of the ground state that also takes into account the $\bar{\Gamma}$ projection for every mass level.

There is of course no factor of $\sqrt{w}$ because the ground state has vanishing number eigenvalue. Performing the trace gives

$$G_{\rm R}(w) = 8 \prod_{n=1}^{\infty} \left(\frac{1 + w^n}{1 - w^n} \right)^8$$

Now, we can address the question of whether there are an equal number of bosons and fermions at each level,

$$G_{\rm NS}(w) = G_{\rm R}(w)$$

Space-time supersymmetry

This is the case: the identity was proved in 1829 by Jacobi who referred to it as "a rather obscure formula" (*aequatio identica satis abstrusa*). It was named after this the abstruse identity. It is not hard to verify that

$$G_{\mathrm{NS}}(w) = G_{\mathrm{R}}(w) = 8(1 + 16\,w + 144\,w^2 + \dots)$$

This observation does not constitute a proof of supersymmetry but, still, it is encouraging.

We will see in what follows that the GSO projection is mandatory in order to build a consistent interacting theory from the RNS superstring.

It is sufficient to work out the 1-loop vacuum amplitude since it does not involve interactions but fully depends on the superstring spectrum.

We will start by computing the bosonic 1-loop closed string vacuum diagram and then we shall extend the result to fermionic world-sheet fields.

The relevant Feynman graph is a torus.