Lecture 8: 1-loop closed string vacuum amplitude

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STRING THEORY

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The vacuum energy of a scalar field

Let us start from the simplest case of a massive scalar field in D dimensions,

$$S = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 \right) .$$

After an Euclidean rotation, the path integral defines the vacuum energy, Z,

$$e^{-Z} = \int \mathcal{D}\phi \, e^{-S_E} = \det{}^{-1/2}(-\Delta + M^2) \quad \Rightarrow \quad Z = \frac{1}{2} \log\det\left(-\Delta + M^2\right)$$

We can parameterize the right hand side as

$$Z = -\frac{1}{2} \int_\epsilon^\infty \frac{dt}{t} \, \mathrm{tr}_{\mathcal{H}} \left(e^{-t\,(-\Delta + M^2)} \right)$$

where ϵ is an UV cutoff and t is a Schwinger parameter.

Using the momentum basis, we can diagonalize the kinetic operator

$$Z = -\frac{V}{2} \int_\epsilon^\infty \frac{dt}{t} \, e^{-t\,M^2} \int \frac{d^D p}{(2\pi)^D} \, e^{-t\,p^2}$$

The vacuum energy of a supersymmetric theory

Performing the Gaussian momentum integral,

$$Z = -\frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}}\ e^{-t\,M^2}$$

If we carry on the same computation for a massive Dirac fermion,

$$Z = \frac{2^{D/2}\ V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}}\ e^{-t\,M^2}$$

due to the Grassmannian nature of the fermionic path integral.

Since Z is only sensitive to the physical modes, proportional to their number,

$$Z = -\frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}}\ \mathrm{Str}_{\mathcal{H}}\, e^{-t\,M^2}$$

Thus, for the bosonic string we should have

$$Z = -\frac{V}{2(4\pi)^{13}} \int_{-1/2}^{1/2} ds \int_\epsilon^\infty \frac{dt}{t^{14}}\ \mathrm{tr}_{\mathcal{H}}\, e^{-\frac{2}{\alpha'}\left(N+\bar{N}-2\right)t+2\pi i\left(N-\bar{N}\right)s}$$

the last term introducing a δ-function constraint: $N = \bar{N}$ (level matching).

1-loop closed (bosonic) string vacuum amplitude

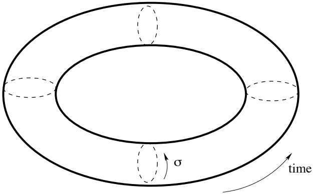
If we define the complex Schwinger parameter,

$$\tau = \tau_1 + i\tau_2 = s + i\frac{t}{\alpha'\pi}$$

and letting $q = e^{2\pi i\tau}$, the previous expression can be rewritten as

$$Z = -\frac{V}{2(4\pi^2\alpha')^{13}} \int_{-1/2}^{1/2} d\tau_1 \int_{\epsilon}^{\infty} \frac{d\tau_2}{\tau_2^{14}} \, \mathrm{tr}_{\mathcal{H}} \left(q^{N-1} \, \bar{q}^{\bar{N}-1} \right)$$

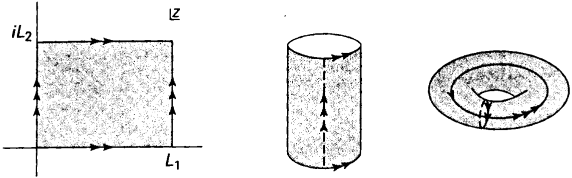
Now, the Feynman graph describing a closed string state which propagates in time and returns back to its initial state is a torus



The rectangular torus

The analytic identifications that lead to the rectangular torus are

$$z \sim z + L_1 \ , \quad z \sim z + L_2 \ .$$



The fundamental domain is $0 \leq \mathrm{Re}\, z < L_1$, $0 \leq \mathrm{Im}\, z < L_2$.

The identification being analytic, the torus is a Riemann surface.

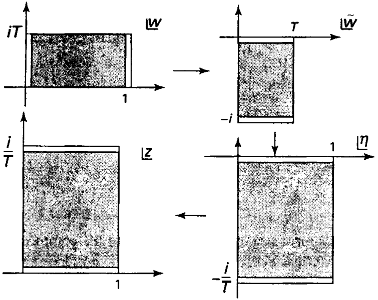
By the trivial conformal map $w = z/L_1$, we get

$$w \sim w + 1 \ , \quad w \sim w + iT \ , \qquad T = L_2/L_1 \ ,$$

the rectangular torus has just one parameter, T.

Rectangular tori with different T can be conformally equivalent.

Consider a torus with $T < 1$. The map $\tilde{w} = -iw$, followed by the scaling $\eta = \tilde{w}/T$ and the lifting up $z = \eta + i/T$:



Tori with parameters T and $1/T$ are conformally equivalent!

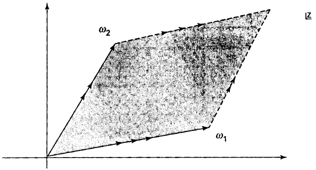
The *moduli space* of rectangular tori can be chosen as $1 \leq T < \infty$.

String amplitudes do not seem to give rise to UV problems!

But tori can be twisted...

The torus

One can glue the bottom and top edges of the cylinder with a twist. Take two (non-parallel) complex numbers w_1 and w_2, such that $\mathrm{Im}(w_2/w_1) > 0$.



The torus is obtained by the identifications, $T^2 = \mathbb{C}/\Lambda_{(w_1,w_2)}$,

$$z \sim z + w_1 \ , \quad z \sim z + w_2 \ ,$$

the fundamental domain being the parallelogram shaded in the figure.

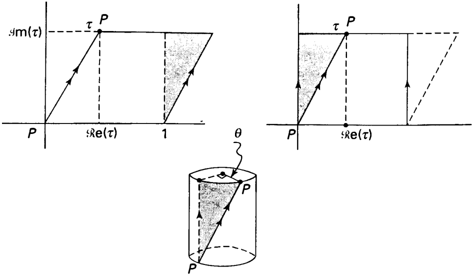
We can rescale $w = z/w_1$, such that

$$\tau \equiv \frac{w_2}{w_1} \ , \qquad \mathrm{Im}\tau > 0 \ ,$$

and the identifications result $w \sim w + 1$ and $w \sim w + \tau$, where $\tau \in \mathbb{H}$.

The torus

Let us see the twitting of tori with $\mathrm{Re}\,\tau \neq 0$. Making use of $w \sim w + 1$, we can drive the parallelogram rectangular.



The identification of horizontal lines is, however, shifted (twisted cylinder).

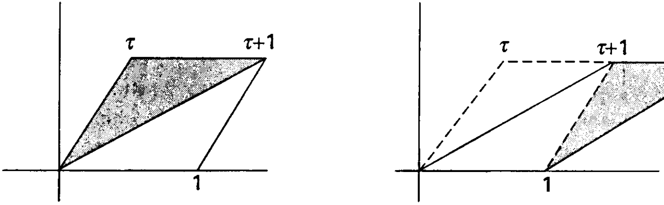
The twist angle, θ, associated with the shift of point P, is

$$\theta = 2\pi\,\mathrm{Re}\,\tau\ .$$

What happens if we let the twist angle to increase by 2π?

The torus

This amounts to $\mathrm{T} : \tau \to \tau + 1$. These tori are in fact the same:



The shift modifies the fundamental region but does not change the torus.

Twist angles beyond $-\pi < \theta \leq \pi$ do not yield new tori.

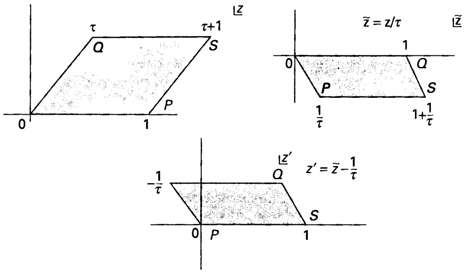
While $\tau \in \mathbb{H}$, the space of inequivalent tori is much smaller. T shows that any infinite vertical strip of unit width is enough,

$$\mathcal{S}_0 \equiv \left\{ -\frac{1}{2} < \mathrm{Re}\, \tau \leq \frac{1}{2} , \quad \mathrm{Im}\, \tau > 0 \right\} .$$

What happens if we perform the transformation $\mathrm{S} : \tau \to -1/\tau$?

The torus

If we first do $\tilde{z} = z/\tau$, which is a rigid rotation plus a uniform scaling, followed by a rigid translation, $z' = \tilde{z} - 1/\tau$,



the final parallelogram is associated with a torus of parameter $-1/\tau$.

The torus is given by a flat metric and a complex structure $\tau \in \mathbb{C}$, living in a fundamental domain that we will shortly determine.

τ specifies the shape of the torus which cannot be changed by conformal transformations or any local change of coordinates.

The torus: modular group

The full family of equivalent tori can be reached by modular transformations, which are combination of the operations T and S,

$$\mathrm{T} : \tau \to \tau + 1 \qquad \mathrm{S} : \tau \to -\frac{1}{\tau} \qquad \mathrm{T\,S\,T} : \tau \to \frac{\tau}{\tau + 1}$$

T and S obey the relations: $\mathrm{S}^2 = (\mathrm{S\,T})^3 = 1$.

They generate the modular group $SL(2, \mathbb{Z})$ of the torus,

$$\tau \;\to\; \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \;,$$

which is a linear fractional transformation of τ.

They correspond to using $\Lambda_{(w_1, w_2)}$ or $\Lambda_{(w_1', w_2')}$, generated by

$$w_1' = a\, w_1 + b\, w_2 \quad w_2' = c\, w_1 + d\, w_2 \qquad a, b, c, d \in \mathbb{Z}$$

with $ad - bc = 1$ (the area of the parallelogram): both lattices are the same.

We should sum only conformally inequivalent tori defining the moduli space

$$\mathcal{M}_{T^2} = \mathcal{H}/PSL(2,\mathbb{Z}) \qquad PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$$

where $\mathcal{H}$ is the upper half plane and $\mathbb{Z}_2$ takes into account of the equivalence of an $SL(2,\mathbb{Z})$ matrix and its negative.

We have to identify a fundamental region, $\mathcal{F}$, of the τ plane, such that any point in $\mathcal{H}$ can be mapped to $\mathcal{F}$ through an element of $SL(2,\mathbb{Z})$:

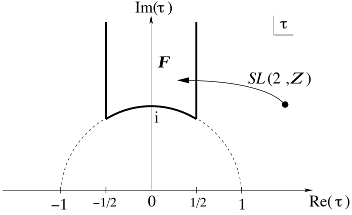


Figure: Here, $\mathcal{F} = \{-\frac{1}{2} \leq \tau_1 \leq 0, |\tau|^2 \geq 1\} \cup \{0 < \tau_1 < \frac{1}{2}, |\tau|^2 > 1\}$. Any point outside $\mathcal{F}$ can be mapped into the interior by a modular transformation.

1-loop closed (bosonic) string vacuum amplitude

We must do the path integral of the 1-loop amplitude by integrating on the fundamental modular domain, $\mathcal{F}$.

As previously mentioned, we start by computing it for the bosonic string,

$$Z = -\frac{V}{2(4\pi^2\alpha')^{13}} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau^2} \, \frac{1}{\mathrm{Im}\,\tau^{12}} \, \mathrm{tr}_{\mathcal{H}} \left(q^{N-1} \, \bar{q}^{\bar{N}-1} \right)$$

Now, recall the generating function that led us to the computation of the degeneracy of states,

$$G(q) := \mathrm{tr}_{\mathcal{H}} \, q^N = \prod_{n=1}^{\infty} (1 - q^n)^{-24} \, .$$

Thus, since $G(q) = q\,\eta(\tau)$, the path integral at 1-loop reads

$$Z = \int_{\mathcal{F}} d^2\tau \frac{1}{\tau_2^{12}} \, \frac{1}{q\bar{q}} \left| \prod_{n=1}^{\infty} (1 - q^n)^{-24} \right|^2 = \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau^2} \, \frac{1}{\mathrm{Im}\,\tau^{12}} \, |\eta(\tau)|^{-48}$$

Modular invariance of the path integral

where we have introduced the Dedekind function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

with the modular transformation properties

$$\eta(\tau + 1) = \eta(\tau)$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\ \eta(\tau)$$

It can be readily shown that Z is modular invariant.

However, when actually computed, Z diverges as a consequence of the tachyonic instability of the bosonic string theory.

How does this generalizes to the case of the RNS superstring?

Fermions on a torus and spin structures

Fermions on the torus are specified by a choice of spin structure which is simply a choice of boundary conditions as $\xi^0 \to \xi^0 + 2\pi$ and $\xi^1 \to \xi^1 + 2\pi$. Notice that this generalizes *easily* to higher genus cases.

There are four possible spin structures in all, which we denote symbolically

$$\xi^0 \overset{\xi^1}{\square} = +\overset{+}{\square}\,,\quad +\overset{-}{\square}\,,\quad -\overset{+}{\square}\,,\quad -\overset{-}{\square}$$

where the squares denote the result of performing the functional integral over fermions with the given fixed spin structure.

Let us focus on the right-moving sector.

The NS (R) sector corresponds to anti-periodic (periodic) boundary conditions along the string in the ξ^1 direction.

In the time direction ξ^0, fermions are anti-periodic, in accordance with the standard Euclidean path integral formulation of finite temperature QFT.

Spin structure contributions to the path integral

In order to flip the boundary condition, we insert the Klein operator $(-1)^F$ (since it anticommutes with ψ^μ) in the traces defining the partition function.

The fermionic spin structure contributions to the path integral are, thereby,

$$+\,\overset{+}{\square} \;\to\; \mathrm{Tr}_{\mathrm{R}}\Big((-1)^F\, e^{-2\pi\tau_2 H}\Big) \qquad -\,\overset{+}{\square} \;\to\; \mathrm{Tr}_{\mathrm{R}}\Big(e^{-2\pi\tau_2 H}\Big)$$

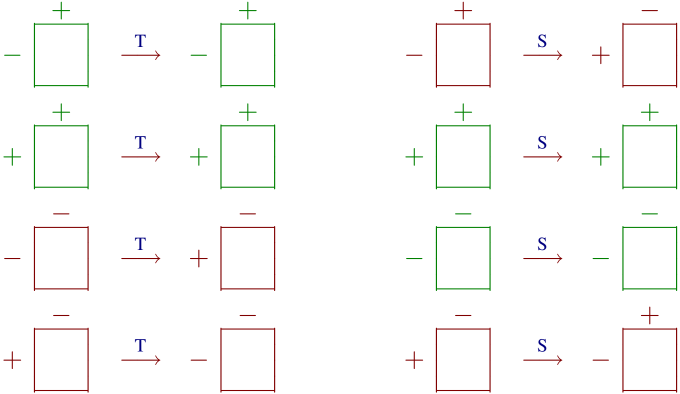
$$-\,\overset{-}{\square} \;\to\; \mathrm{Tr}_{\mathrm{NS}}\Big(e^{-2\pi\tau_2 H}\Big) \qquad +\,\overset{-}{\square} \;\to\; \mathrm{Tr}_{\mathrm{NS}}\Big((-1)^F\, e^{-2\pi\tau_2 H}\Big)$$

But the modular group in general changes the femionic boundary conditions. We know that the final result, since it involves a partition function on a torus, must be modular invariant.

Let us study the action of the modular group on the different spin structures. We can focus on the operations T and S.

Modular invariance of spin structures

T: $\tau \to \tau + 1$ induces an additional periodic shift from ξ^1 along ξ^0. It therefore flips the ξ^0 boundary condition whenever the ξ^1 is anti-periodic.

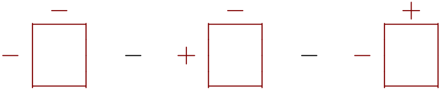


S: $\tau \to -1/\tau$ induces a switch $(\xi^0, \xi^1) \to (-\xi^1, \xi^0)$. Thus, both periodicities are simply exchanged. The $(+,+)$ spin structure is modular invariant.

The 1-loop partition function for the RNS superstring

The amplitude corresponding to $(+,+)$ vanishes: it contains a Grassmann integral over the constant fermionic zero-modes but H is independent of them.

This is the only amplitude with fermionic zero-modes. For the remaining spin structures, the unique modular invariant combination is given by



The 1-loop, modular invariant partition function for the right-moving fermionic contributions is therefore given by

$$Z_{\mathrm{RNS}} = \frac{1}{2}\,\mathrm{Tr}_{\mathrm{NS}}\left[\left(1-(-1)^F\right) q^{L_0-\frac{1}{2}}\right] - \frac{1}{2}\,\mathrm{Tr}_{\mathrm{R}}\left(q^{L_0}\right)$$

Adding the vanishing contribution of $(+,+)$, the amplitude can be succinctly as a trace over the full right-moving fermionic Hilbert space as

$$Z_{\mathrm{RNS}} = \mathrm{Tr}_{\mathrm{NS}\oplus\mathrm{R}}\left(P^{\pm}_{\mathrm{GSO}}\, q^{L_0-a}\right)$$

Modular invariance

We arrived at a beautiful interpretation of the GSO projection, which ensures vacuum stability and spacetime supersymmetry of the quantum string theory,

$$Z_{\mathrm{RNS}} = \mathrm{Tr}_{\mathrm{NS}\oplus\mathrm{R}} \left(P^{\pm}_{\mathrm{GSO}}\, q^{L_0 - a} \right)$$

Geometrically, it is simply the *modular invariant* sum over spin structures. This interpretation also generalizes to higher-loop amplitudes.

The total superstring amplitude is given by the product of Z_{RNS} with Z_{bos} in ten dimensions, along with their left-moving counterparts.

It is an exercise reminiscent of the computation of the spectrum degeneracy that finally shows that the 1-loop closed RNS superstring amplitude vanishes thanks to Jacobi.

This implies that the full superstring vacuum amplitude vanishes which provides very strong evidence in favor of space-time supersymmetry in string theory in ten dimensions.

The Ramond-Ramond sector

Recall that for open strings there are two possible R sector GSO projections given by the operators

$$P^{\pm}_{\mathrm{GSO}} := \frac{1}{2}\left(1 \pm \Gamma^{11}\,(-1)^{\sum_{r=1}^{\infty}\psi_{-r}\cdot\psi_r}\right)$$

the sign choice selecting a Ramond ground state with $\pm$ chirality,

$$\Gamma_{11}\,|\psi^{(0)}_{(\pm)}\rangle_{\mathrm{R}} = \pm|\psi^{(0)}_{(\pm)}\rangle_{\mathrm{R}}$$

as well as the corresponding massive tower of states. Both chiralities, though, lead to the same physics.

In the closed string sector the situation is subtler since left- and right-movers are combined together with their particular choices of chiralities. The GSO projection is performed separately in both sectors.

Depending on the relative sign, there are two inequivalent possibilities that correspond to the relative chirality of the ground states.

Type II suprestring theories

There are, thus, two possible string theories that we can construct in this way, Type IIA and Type IIB:

- Type IIA: We take the *opposite* GSO projection on both sides; the spinors are of opposite chirality,

  $$|\psi^{(0)}_{+(+)}\rangle_{\rm R} \qquad |\psi^{(0)}_{-(-)}\rangle_{\rm R}$$

  and hence construct the whole R-R sector of the spectrum by further action of the creation operators on these states. The resulting theory is non-chiral.

- Type IIB: We impose the *same* GSO projection on both sides,

  $$|\psi^{(0)}_{+(+)}\rangle_{\rm R} \qquad |\psi^{(0)}_{-(+)}\rangle_{\rm R}$$

  The R-R sector of the spectrum by further action of the creation operators on these states. This leads to a chiral theory.

Massless spectrum in the RR-sector of type II superstring theories

As discussed earlier, the two irreducible, Majorana-Weyl representations of $\mathrm{Spin}(10)$ are the spinor $\mathbf{16}_s$ and its conjugate $\mathbf{16}_c$.

From a group theoretic perspective then, the massless states of the two R-R sectors are characterized by their Clebsch-Gordan decompositions:

- Type IIA: $\mathbf{16}_s \otimes \mathbf{16}_c = [0] \oplus [2] \oplus [4]$.
- Type IIB: $\mathbf{16}_s \otimes \mathbf{16}_s = [1] \oplus [3] \oplus [5]_+$.

Here $[n]$ denotes the irreducible n-times antisymmetrized representation of $Spin(10)$ given by $\bar{\psi}\,\Gamma^{\mu_1\cdots\mu_n}\,\psi$, where ψ is a spinor in the relevant irreducible representation.

They corresponds to completely antisymmetric tensors of rank n, or n-forms. The $+$ subscript indicates a self-duality condition.

Other critical string theories

- Type I: Can be obtained from a projection of the Type IIB that keeps only the diagonal sum of the two gravitinos Ψ_μ and Ψ'_μ.

  It has only $\mathcal{N} = 1$ space-time supersymmetry.

  It is a theory of unoriented string world-sheets.

  In the open string sector, anomaly cancellation singles out $SO(32)$ as the only possible (Chan-Paton) gauge group.

- Heterotic string theory: It comprises a heterosis of the $d = 26$ bosonic string for the left-movers and the $d = 10$ superstring for the right-movers.

  The remaining 16 right-moving degrees of freedom required by $N = 1$ supersymmetry are "internal" ones which come from a 16-dimensional, modular invariant lattices.

  There are only two such lattices, corresponding to the weight lattices of the Lie groups $E_8 \times E_8$ and $SO(32)$.

There is a total of five critical string theories.