Lecture 9: RR-sector and D-branes

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STRING THEORY

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RR fields couple to extended objects

Recall that the antisymmetric tensor $B_{\mu\nu}$ in the NS-NS sector couples directly to the string world-sheet: the string carries (electric) *charge* w.r.t. $B_{\mu\nu}$,

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \, \epsilon^{\alpha\beta} \, B_{\mu\nu}(X) \, \partial_\alpha X^\mu \, \partial_\beta X^\nu \ .$$

The Lagrangian changes by a total derivative under the gauge symmetry,

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

In electromagnetism, the gauge invariant degrees of freedom are contained in the field strength, $F = d\,A$. Similarly, $H = d\,B$,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

However, the situation for the R-R potentials $C^{(n)}$ is very different, because the vertex operators for the R-R states involve only the $F^{(n+1)}$.

Thus, only the field strengths, not the potentials, would couple to the string.

Thus elementary, perturbative string states cannot carry any charge with respect to the R-R gauge fields $C^{(n)}$.

RR fields couple to extended objects

We are thereby forced to search for non-perturbative degrees of freedom which couple to these potentials.

Clearly, they must be extended objects that sweep out a $p+1$-dimensional *world-volume* as they propagate in time, generalizing the notion of a string,

$$q \int_{\mathcal{W}_n} d^n\xi \; \epsilon^{a_0\cdots a_{n-1}} \frac{\partial x^{\mu_1}}{\partial \xi^{a_0}} \cdots \frac{\partial x^{\mu_n}}{\partial \xi^{a_{n-1}}} \; C^{(n)}{}_{\mu_1\cdots\mu_n} \quad \longrightarrow \quad q \int_{\mathcal{W}_n} C^{(n)} \, .$$

in complete analogy with electromagnetic and B-field minimal couplings.

The massless states of the R-R sectors are given by the CG decompositions:

- Type IIA: $\mathbf{16}_s \otimes \mathbf{16}_c = [0] \oplus [2] \oplus [4]$.
- Type IIB: $\mathbf{16}_s \otimes \mathbf{16}_s = [1] \oplus [3] \oplus [5]_+$.

They corresponds to completely antisymmetric tensors of rank n, or n-forms. The $+$ subscript indicates a self-duality condition.

Electric and magnetic coupling of extended objects

Consider $F^{(p+2)}$, a $p+2$-form representing an antisymmetric tensor field with $p+2$ indices in D dimensions.

It is the field strength of a potential, $F^{(p+2)} = dC^{(p+1)}$, that *electrically* couples to a $p+1$-dimensional object, $\mathcal{W}_{p+1}$,

$$\mu_p \int_{\mathcal{W}_{p+1}} C^{(p+1)} ,$$

$\mathcal{W}_{p+1}$ being the *world-volume* of an extended object called p-brane.

The Hodge dual of $F^{(p+2)}$ is

$$\tilde{F}^{(D-p-2)} = {}^{\star}F^{(p+2)} = d\tilde{C}^{(D-p-3)} .$$

Its potential couples *magnetically* to the extended object $\mathcal{W}_{D-p-3}$,

$$\mu_{D-p-4} \int_{\mathcal{W}_{D-p-3}} \tilde{C}^{(D-p-3)} .$$

Extended objects in type II superstring theory

In superstring theory, $D = 10$ thus we have two possible couplings

$$F^{(n+2)} \quad \text{couples to} \quad \begin{cases} \text{electric} & n \text{ branes} \,, \\ \text{magnetic} & 6-n \text{ branes} \,. \end{cases}$$

Recall that the $F^{(n)}$ forms resulted from the tensor product of two MW spinor representations in ten dimensions,

$$F^{(n)}_{\mu_1\cdots\mu_n} = {}_{\mathrm{R}}\langle\psi^{(0)}_{+(\pm)}|\Gamma_{[\mu_1\cdots\mu_n]}|\psi^{(0)}_{-(\pm)}\rangle_{\mathrm{R}} \,.$$

Because of GSO projection, the states $|\psi^{(0)}_{-(\pm)}\rangle_{\mathrm{R}}$ have definite Γ^{11} eigenvalue ± 1. Thus, given that

$$\Gamma^{11}\,\Gamma^{[\mu_1}\ldots\Gamma^{\mu_n]} = \frac{(-1)^{\left[\frac{n}{2}\right]}\,n!}{(10-n)!^2}\,\epsilon^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_{10-n}}\,\Gamma^{[\nu_1}\ldots\Gamma^{\nu_{10-n}]} \,,$$

there is an isomorphism (*electric-magnetic duality*)

$$F^{(n)}_{\mu_1\cdots\mu_n} \sim \epsilon^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_{10-n}}\,F^{(10-n)}_{\nu_1\cdots\nu_{10-n}} \,.$$

This identifies the representations $[n] \leftrightarrow [10-n]$; in particular, $[5]$ is self-dual.

Branes in type IIA theory

Recall that, in type IIA: $\mathbf{16}_s \otimes \mathbf{16}_c = [0] \oplus [2] \oplus [4]$. Thus, there are even branes.

- In the NS-NS sector:

$$H_3 \quad \text{couples to} \quad \begin{cases} \text{electric } 1 \text{ branes} & \Rightarrow \quad \text{F1-string} \\ \text{magnetic } 5 \text{ branes} & \Rightarrow \quad \text{NS5-brane} \end{cases}$$

- In the R-R sector:

$$F_{[2]} \quad \text{couples to} \quad \begin{cases} \text{electric } 0 \text{ branes} & \Rightarrow \quad \text{D0-brane} \\ \text{magnetic } 6 \text{ branes} & \Rightarrow \quad \text{D6-brane} \end{cases}$$

$$F_{[4]} \quad \text{couples to} \quad \begin{cases} \text{electric } 2 \text{ branes} & \Rightarrow \quad \text{D2-brane} \\ \text{magnetic } 4 \text{ branes} & \Rightarrow \quad \text{D4-brane} \end{cases}$$

Branes in type IIB theory

In type IIB: $\mathbf{16}_s \otimes \mathbf{16}_s = [1] \oplus [3] \oplus [5]_+$. Thus, there are odd branes.

- In the NS-NS sector:

$$H_3 \quad \text{couples to} \quad \begin{cases} \text{electric } \ 1 \text{ branes} & \Rightarrow \quad \text{F1-string} \\ \text{magnetic } \ 5 \text{ branes} & \Rightarrow \quad \text{NS5-brane} \end{cases}$$

- In the R-R sector:

$$F_{[1]} \quad \text{couples to} \quad \begin{cases} \text{electric } \ -1 \text{ branes} & \Rightarrow \quad \text{D(-1)-brane} \\ \text{magnetic } \ 7 \text{ branes} & \Rightarrow \quad \text{D7-brane} \end{cases}$$

$$F_{[3]} \quad \text{couples to} \quad \begin{cases} \text{electric } \ 1 \text{ branes} & \Rightarrow \quad \text{D1-brane} \\ \text{magnetic } \ 5 \text{ branes} & \Rightarrow \quad \text{D5-brane} \end{cases}$$

$$F_{[5]} \quad \text{couples to} \quad \begin{cases} \text{electric } \ 3 \text{ branes} & \Rightarrow \quad \text{D3-brane} \\ \text{magnetic } \ 3 \text{ branes} & \Rightarrow \quad \text{D3-brane} \end{cases}$$

Since the $F_{[5]}$ is self-dual, full electromagnetic duality is in place.

Electric-magnetic duality

Let us recall how the story goes in Maxwell theory. In the absence of charges and currents,

$$d\,F = 0 \qquad \text{and} \qquad d\,{}^\star F = 0\,,$$

where F is the 2-form field strength describing electric and magnetic fields.

The equations are symmetric under the interchange of F and ${}^\star F$. Assuming that sources can be added in a symmetric fashion,

$$d\,F = {}^\star J_m \qquad \text{and} \qquad d\,{}^\star F = {}^\star J_e\,,$$

we face Dirac's quantization condition: the wave function of an electrically charged particle moving in the field of a monopole is uniquely defined if

$$e \cdot g \in 2\pi\,\mathbb{Z}\,.$$

We have seen that our branes, D-branes, can both couple electrically or magnetically. Their charges are measured using Gauss' law.

The Dirac quantization condition has a straightforward generalization,

$$\mu_p \cdot \mu_{6-p} \in 2\pi\,\mathbb{Z}\,.$$

Type II low energy effective actions

Similarly to the bosonic case, the vanishing of the Weyl anomaly demands,

$$\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta_{\mu\nu}(\Phi) = 0 \ ,$$

where these equations are covariant complicated expressions of the massless fields.

In type II superstrings we have, in addition, we should include the RR-forms in a way compatible with supersymmetry.

Thus, these equations coincide with those arising in ten dimensional theories of supergravity.

The number of supersymmetries is 32. Whereas type IIB theory is chiral, type IIA is not. We will present their Lagrangians next.

Higher order α' corrections would lead to higher powers of the curvature, as in the bosonic string.

Type IIA effective action

The Lagrangian of type IIA supergravity in the string frame reads

$$S_{IIA} = \frac{1}{2\kappa^2} \int e^{-2\Phi} \Big(d^{10}x \sqrt{-g}\, R + 4\, d\Phi \wedge \star d\Phi - \frac{1}{2} \frac{1}{3!} H_{[3]} \wedge \star H_{[3]}$$
$$- \frac{1}{2} \Big[\frac{1}{2!} F_{[2]} \wedge \star F_{[2]} + \frac{1}{4!} F_{[4]} \wedge \star F_{[4]} + B_{[2]} \wedge F_{[4]} \wedge F_{[4]} \Big] \Big) ,$$

where

$$F_{[2]} = dC_{[1]} \qquad F_{[4]} = dC_{[3]} - C_{[1]} \wedge H_{[3]} \qquad H_{[3]} = dB_{[2]}$$

We can go to the Einstein frame by

$$(g_{\mu\nu})_{\text{string}} = g_s^{-1/2}\, e^{\Phi/2}\, (g_{\mu\nu})_{\text{Einstein}}$$

where $g_s = e^{\Phi(r\to\infty)}$ is the string coupling constant. Then,

$$\sqrt{|g|}_{\text{string}} = (g_s^{-1/2}\, e^{\Phi/2})^5 \sqrt{|g|}_{\text{Einstein}} = g_s^{-5/2}\, e^{5\Phi/2} \sqrt{|g|}_{\text{Einstein}} .$$

Type IIA effective action

In general, for any p–form

$$(F_{[p]}\wedge{}^{\star}F_{[p]})_{\text{string}} = g_s^{p/2}\, e^{-p\,\Phi/2}\,(F_{[p]}\wedge{}^{\star}F_{[p]})_{\text{Einstein}}\ .$$

The resulting action in the Einstein frame reads:

$$\begin{aligned} S^E_{IIA} \;=\; & \frac{1}{2\kappa^2}\int\Big(d^{10}x\,\sqrt{-g_E}\;R_E - \frac{1}{2}\,d\Phi\wedge\star d\Phi - \frac{1}{2}\,\frac{1}{3!}\,e^{-\Phi}\,H_{[3]}\wedge\star H_{[3]} \\ & -\frac{1}{2}\Big[\frac{1}{2!}\,e^{\frac{3}{2}\Phi}\,F_{[2]}\wedge\star F_{[2]} + \frac{1}{4!}\,e^{\frac{1}{2}\Phi}\,F_{[4]}\wedge\star F_{[4]} + B_{[2]}\wedge F_{[4]}\wedge F_{[4]}\Big]\Big)\ . \end{aligned}$$

Gravity is now canonically normalized, as well as the dilaton kinetic term, but the coupling with the R-R forms is more involved.

Notice that to compute the solution corresponding to a specific D-brane,

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\,\Big\{R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}e^{a_n\,\phi}\,{F_n}^2\Big\}\ .$$

Type IIB effective action

The Lagrangian of type IIB supergravity in the string frame reads

$$
\begin{aligned}
S_{IIB} \;=\; & \frac{1}{2\kappa^2}\int e^{-2\Phi}\left(d^{10}x\sqrt{-g}\;R + 4\;d\Phi\wedge\star d\Phi - \frac{1}{2\,3!}H_{[3]}\wedge\star H_{[3]}\right) \\
& -\frac{1}{4\kappa^2}\int\left(F_{[1]}\wedge\star F_{[1]} + \frac{1}{3!}\,F_{[3]}\wedge\star F_{[3]} + \frac{1}{2}\,\frac{1}{5!}\,F_{[5]}\wedge\star F_{[5]}\right. \\
& \left. -C_{[4]}\wedge F_{[3]}\wedge H_{[3]}\right)
\end{aligned}
$$

where

$$
F_{[1]} = d\,C_{[0]} \qquad F_{[3]} = d\,C_{[2]} - C_{[0]}\,H_{[3]}
$$

$$
H_{[3]} = d\,B_{[2]} \qquad F_{[5]} = d\,C_{[4]} - \frac{1}{2}\,C_{[2]}\wedge H_{[3]} + \frac{1}{2}\,B_{[2]}\wedge F_{[3]}
$$

supplemented by the additional on-shell constraint $F_{[5]} = \star F_{[5]}$.

Type IIB effective action

It can also be driven to its Einstein frame form,

$$\begin{aligned} S_{IIB}^{E} \;=\; \frac{1}{2\kappa^2}\int \Bigg(& d^{10}x\,\sqrt{-g_E}\,R_E - \frac{1}{2}\,d\Phi\wedge\star d\Phi - \frac{1}{2\,3!}e^{-\Phi}\,H_{[3]}\wedge\star H_{[3]} \\ & -\frac{1}{2}\left[e^{2\Phi}\,F_{[1]}\wedge\star F_{[1]} + \frac{1}{3!}\,e^{\Phi}\,F_{[3]}\wedge\star F_{[3]} + \frac{1}{2}\,\frac{1}{5!}\,F_{[5]}\wedge\star F_{[5]}\right] \\ & -C_{[4]}\wedge F_{[3]}\wedge H_{[3]}\Bigg)\,. \end{aligned}$$

Again, to compute the solution corresponding to a specific D-brane,

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left\{R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}e^{a_n\,\phi}\,F_n{}^2\right\}\,.$$

We have $a_3 = -1$ for the NS-NS 3-form, $H_{[3]}$, and $a_n = \frac{5-n}{2}$ for any RR n-form, $F_{[n]}$. Notice that for $n = 5$, *i.e.*, the self-dual D3-brane, the dilaton decouples.

Even the remaining string theories fit into this quite simple action.

D-branes as classical solutions

The equations of motion are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \left(\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\lambda \phi \, \partial^\lambda \phi \right) + \frac{1}{2(n-1)!} \, e^{a_n \phi} \, \tau_{\mu\nu} \ ,$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \, g^{\mu\nu} \, \partial_\nu \phi \right) - \frac{1}{2} \frac{a_n}{n!} \, e^{a_n \phi} \, F_n^2 = 0 \ ,$$

$$\frac{1}{(n-1)!} \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \, e^{a_n \phi} \, F^{\mu\nu_2 \ldots \nu_n} \right) = 0 \ ,$$

where the electromagnetic stress-energy tensor reads

$$\tau_{\mu\nu} = F_{\mu\lambda_2 \ldots \lambda_{n_l}} \, F_\nu{}^{\lambda_2 \ldots \lambda_{n_l}} - \frac{1}{2n} \, g_{\mu\nu} \, F_n^2 \ .$$

The most general metric that incorporates all the symmetries is:

$$ds^2 = -B^2 dt^2 + C^2 \delta_{ij} \, dy^i dy^j + F^2 \, dr^2 + G^2 \, r^2 \, d\Omega_{d-1}^2 \ .$$

with all the functions depending only on r.

D-branes as classical solutions

There is an extremal solution

$$ds^2 = H^{-\frac{2(7-p)}{\Delta}} \left(-dt^2 + \delta_{ij}\, dy^i dy^j\right) + H^{\frac{2(p+1)}{\Delta}} \left(dr^2 + r^2\, d\Omega_{d-1}^2\right) .$$

with $\Delta = (p+1)(7-p) + 4\, a_n^2$ and

$$H = 1 + \frac{\sqrt{\Delta}}{4(7-p)}\, \frac{Q}{r^{7-p}} .$$

Now, electric solutions are those with $p = n-2$ whereas magnetic solutions have $p = 8-n$. The dilaton reads

$$e^{\phi} = H^{\frac{8\, a_{p+2}}{\Delta}} \qquad \text{or} \qquad e^{\phi} = H^{-\frac{8\, a_{8-p}}{\Delta}} ,$$

whereas the R-R form

$$F_{ty^1\cdots y^p r} = \frac{d}{dr} H^{-1} \qquad \text{or} \qquad F_{\theta_1\cdots\theta_{8-p}} = Q\, \omega_{8-p} ,$$

the latter being proportional to the volume element of S^{8-p}.

The D3-brane

The solutions look simpler in the string frame,

$$ds^2 = H^{-\frac{1}{2}} \left(-dt^2 + \delta_{ij}\, dy^i dy^j\right) + H^{\frac{1}{2}} \left(dr^2 + r^2\, d\Omega_5^2\right) .$$

for any p. The mass of these solutions can be computed

$$M = |q| \qquad\qquad q = \frac{L^p\, \Omega^{8-p}}{2\kappa^2}\, Q \,;$$

they all saturate the BPS bound.

As mentioned earlier, the case $n = 5$ (that is, $p = 3$) is special. If we plug it into previous expressions, $a_5 = 0$, $\Delta = 16$, and

$$ds^2 = \left(1 + \frac{1}{4}\,\frac{Q}{r^4}\right)^{-\frac{1}{2}} \left(-dt^2 + \delta_{ij}\, dy^i dy^j\right) + \left(1 + \frac{1}{4}\,\frac{Q}{r^4}\right)^{\frac{1}{2}} \left(dr^2 + r^2\, d\Omega_5^2\right) .$$

If we focus in the region close to the throat, $r \to 0$, the metric behaves as

$$ds^2 \sim r^2 \left(-dt^2 + \delta_{ij}\, dy^i dy^j\right) + r^{-2} \left(dr^2 + r^2\, d\Omega_5^2\right) .$$

This is $\mathrm{AdS}_5 \times \mathrm{S}^5$ with equal radii of curvature.

D-branes and boundary contributions

Let us come back to the gauge symmetry of (we set $2\pi\alpha' = 1$)

$$S = -\frac{1}{2}\int d^2\xi\,\epsilon^{\alpha\beta}\,B_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\ .$$

If we perform the gauge symmetry transformation,

$$\delta B_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$$

then the action transforms as

$$\delta S = -\int d^2\xi\,\epsilon^{\alpha\beta}\,\partial_\mu\Lambda_\nu\,\partial_\alpha X^\mu\,\partial_\beta X^\nu = -\int d^2\xi\,\epsilon^{\alpha\beta}\,\partial_\alpha\Lambda_\nu\,\partial_\beta X^\nu\ .$$

$$= -\int d\tau\,d\sigma\,\left(\partial_\tau\Lambda_\nu\,\partial_\sigma X^\nu - \partial_\sigma\Lambda_\nu\,\partial_\tau X^\nu\right)$$

$$= -\int d\tau\,d\sigma\,\left(\partial_\tau\left[\Lambda_\nu\,\partial_\sigma X^\nu\right] - \partial_\sigma\left[\Lambda_\nu\,\partial_\tau X^\nu\right]\right)$$

the total derivative ∂_τ gives no boundary contribution but ∂_σ does!

D-branes and boundary contributions

From the point of view of the open strings, the D-branes are hypersurfaces where their end-points can lie,

$$\delta S = \int d\tau \, d\sigma \, \left(\partial_\sigma \left[\Lambda_\nu \, \partial_\tau X^\nu\right]\right) = \int d\tau \, \left[\Lambda_\nu \, \partial_\tau X^\nu\right] \Big|_{\sigma=0}^{\sigma=\pi}$$

Now, we have to distinguish between $X^\mu = (X^m, X^a)$, where $m = 0, 1, \ldots, p$,

$$\delta S = \int d\tau \, \left[\Lambda_m \, \partial_\tau X^m + \Lambda_a \, \partial_\tau X^a\right] \Big|_{\sigma=0}^{\sigma=\pi} = \int d\tau \, \left[\Lambda_m \, \partial_\tau X^m\right] \Big|_{\sigma=0}^{\sigma=\pi}$$

since $\partial_\tau X^a = 0$ at both end-points.

Gauge invariance fails at the end-points of the string! To restore it, we must add a couple of terms that give electric charge to the string end-points,

$$S = -\frac{1}{2} \int d^2\xi \, \epsilon^{\alpha\beta} \, B_{\mu\nu}(X) \, \partial_\alpha X^\mu \, \partial_\beta X^\nu + \int d\tau \, A_m(X) \, \frac{dX^m}{d\tau} \Big|_{\sigma=0}^{\sigma=\pi} .$$

The string end-points are oppositely charged and $F_{mn} \to \mathcal{F}_{mn} = F_{mn} + B_{mn}$.